RESTRICTED OPEN BOOK EXAMINATION
Data provided: Table of χ^2 quantiles

MAS371

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Applied Probability **2 hours**

Restricted Open Book Examination.
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator which conforms to University regulations.
Marks will be awarded for your best ***three*** *answers. Total marks 60.*

1 The movement of a seal while diving can be described at any instant as Descending, Horizontal (e.g. while feeding) or Ascending. A sensor records the type of movement every minute.

(i) In a single dive, the following sequence of movement behaviours is recorded:

D, D, D, D, D, H, H, H, A, A, H, H, D, H, A, A, A, A.

Assuming that the sequence of behaviours follows a (discrete time) Markov chain, calculate maximum likelihood estimates of each of the transition probabilities $p_{DD}, p_{DH}, \ldots$ ***(4 marks)***

(ii) In a larger study that aggregates data from several dives, the following counts of transitions are obtained.

		Behaviour after $t+1$ minutes		
		Descending	Horizontal	Ascending
Behaviour	Descending	42	12	0
after t	Horizontal	5	15	6
minutes	Ascending	0	8	35

Two possible models are suggested for the transition matrix. The general model has

$$P = \begin{pmatrix} p_{DD} & p_{DH} & 0 \\ p_{HD} & p_{HH} & p_{HA} \\ 0 & p_{AH} & p_{AA} \end{pmatrix},$$

while the symmetric model has

$$P = \begin{pmatrix} p_M & p_S & 0 \\ \frac{1}{2}p_R & p_F & \frac{1}{2}p_R \\ 0 & p_S & p_M \end{pmatrix},$$

where the subscripts stand for Moving, Stopping, Resuming and Feeding.

(a) Assuming the general model, calculate approximate 95% confidence intervals for p_{HH} and for p_{HA}. ***(3 marks)***

(b) Test the null hypothesis that the symmetric model holds against the alternative that the general model holds. You may use the fact that $\sum n_{ij} \log(\hat{p}_{ij}) = -73.67$, where the n_{ij}s represent the counts in the data, the $\hat{p}_{ij}$s represent the maximum likelihood estimates under the general model, and the summation is over elements with $n_{ij} > 0$.

(8 marks)

(iii) Another study reports the following table of transition counts.

		Behaviour after $t+1$ minutes		
		Descending	Horizontal	Ascending
Behaviour	Descending	33	9	0
after t	Horizontal	7	13	7
minutes	Ascending	0	6	22

Without detailed calculation, explain what these data suggest about the null hypothesis from (ii)(b), and about the appropriateness of these models more generally. ***(5 marks)***

2 A Markov chain with states $1, 2, \ldots, s$ has transition matrix P with elements $p_{ij}, i, j = 1, \ldots, s$. A sequence of states of the chain $X_0, X_1, X_2, \ldots, X_N$ is observed.

(i) Explain briefly:

(a) why the log-likelihood for P, conditioning as usual on X_0, is given by

$$\sum_{i,j} N_{ij} \log p_{ij}$$

where N_{ij} is the observed number of transitions from state i to state j; ***(2 marks)***

(b) what constraints must be satisfied by the elements p_{ij} of the transition matrix. ***(2 marks)***

(ii) (a) Show that the log-likelihood for P can be rewritten in terms of the $s(s-1)$ parameters $\{p_{ij}, i \neq j\}$ as

$$l(P|X_0, \ldots, X_N) = \sum_{i,j:i\neq j} N_{ij} \log p_{ij} + \sum_{i} N_{ii} \log(1 - \sum_{k:k\neq i} p_{ik}).$$

(2 marks)

(b) Hence derive the form of the corresponding $s(s-1) \times s(s-1)$ observed information matrix J. ***(6 marks)***

Hint: writing $J_{ij,gh}$ for the element of J corresponding to p_{ij} and p_{gh}, you may wish to consider separately the following cases: $J_{ij,gh}, g \neq i$; $J_{ij,ih}, h \neq j$; and $J_{ij,ij}$.

(iii) For a particular experiment with $s = 3$, the numerical value of $J(\hat{P})$, using the approach in (ii), is found to be

$$\begin{pmatrix} 144.4 & 44.4 & 0.0 & 0.0 & 0.0 & 0.0 \\ 44.4 & 101.6 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 225.6 & 45.1 & 0.0 & 0.0 \\ 0.0 & 0.0 & 45.1 & 85.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 252.0 & 220.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 220.5 & 308.7 \end{pmatrix},$$

where the order of the rows and columns is $p_{12}, p_{13}, p_{21}, p_{23}, p_{31}, p_{32}$. Find the estimated standard error for $\hat{p}_{12}$ in this case. ***(5 marks)***

(iv) Explain briefly how the approach in (ii) can be used to obtain the variance for a diagonal element of the transition matrix P, and its covariances with other elements, *either* for the general case in (ii) or by using the numerical example in (iii). ***(3 marks)***

3 (i) A laboratory experiment observes the times at which cells split into two, in a small population starting with a single cell. The times (in minutes) from the start of the experiment, at which the population increased due to a cell splitting, are as follows:

6.2, 10.3, 12.3, 14.3, 15.5, 16.9, 18.0, 18.9.

Observation stopped after 20 minutes. Assuming that the population can be modelled by a linear birth process, calculate an estimate and standard error for the 'birth' rate per cell per minute. (You may use results from the notes without derivation, provided you state them clearly.) ***(6 marks)***

(ii) The processing of jobs by a multi-node computer system can be modelled as a queue in continuous time, as follows. Jobs arrive at a constant rate α. If there are 2 or more jobs in the system, up to a maximum of k, each one is processed by a single node with individual service rate β. If there are more than k jobs, then k of them will be processed (giving a total service rate $k\beta$) while the rest wait, and there is no practical limit to the number that may be waiting. If there is exactly one job in the system, it is processed using all available computing resources linked together, with an enhanced service rate $\beta\gamma$ with $\gamma > 1$.

(a) Explain why this model can be thought of as a birth-death process, giving expressions for the birth and death rates. ***(3 marks)***

(b) What condition must the parameters satisfy if the number of jobs waiting is not to increase without limit? ***(2 marks)***

(c) Write down the log-likelihood for α, β, γ based on complete observation of the system over an interval $[0, t]$. (You may use standard results for the likelihood of a birth-death process.)

Hence, show that the observations can be reduced to the following sufficient statistics: the total number of arrivals; the total number of completions of service; n_{10}, the number of completions that occured while the system was in 'enhanced' mode (i.e. with exactly one job in the system); a_1, the time spent in 'enhanced' mode; the proportion of time for which there was at least one job waiting but not being processed (i.e. more than k jobs in the system); and the average number of jobs in the system when there were no jobs waiting.

(9 marks)

4 Figure 1 shows the locations of $n = 106$ trees in a 100m square plot.

(i) Assuming a homogeneous Poisson process with intensity λ for the locations, find the maximum likelihood estimate and estimated standard error for λ.

(3 marks)

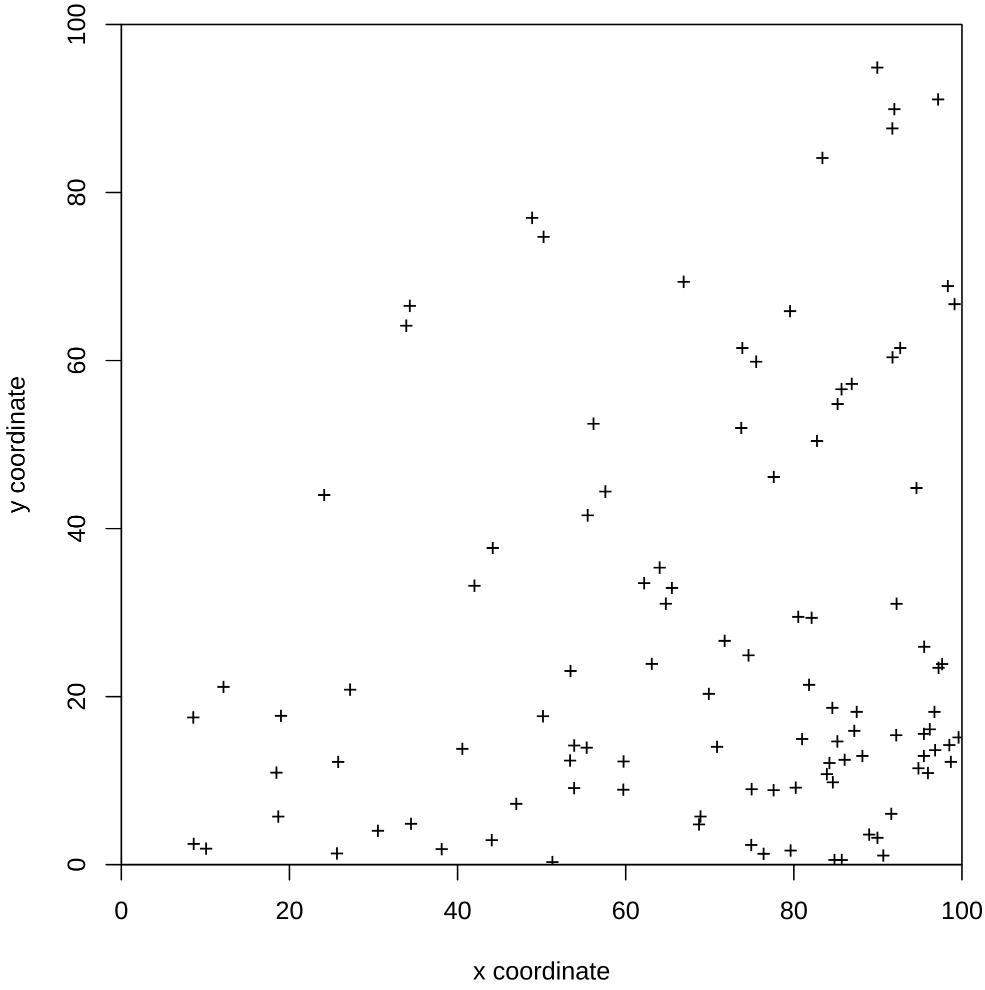

Figure 1: Locations of trees

 Question 4 continued on next page

4 (continued)

(ii) A possible model for the locations is an inhomogeneous Poisson process with intensity function

$$\begin{aligned}\lambda(x,y) &= ab^x c^y \\ &= \exp(\alpha)\exp(\beta x)\exp(\gamma y)\end{aligned}$$

say, where x, y are co-ordinates relative to the south-west corner of the plot, to the east and north respectively, measured in metres (so $0 \leq x, y \leq 100$).

Assuming a model of this form, write down the log-likelihood function for the parameters α, β, γ based on the locations in the figure, and hence the likelihood equations that the maximum likelihood estimators of these parameters must satisfy. (You are not required to derive the form of the estimators.) ***(9 marks)***

Numerical maximisation gives estimates $\hat{\alpha} = -4.88$, $\hat{\beta} = 0.0256$, $\hat{\gamma} = -0.0318$, with a maximised log-likelihood of -527.4. Explain to what extent these parameters are consistent with the visual impression given by the point pattern in Figure 1. ***(3 marks)***

(iii) Further numerical maximisation of the likelihood from (ii), under the restriction that $\beta = 0$, gives a maximum at $\tilde{\alpha} = -3.35, \tilde{\gamma} = -0.0318$, with a maximised log-likelihood of -552.2. Explain the meaning of this restriction on the model, and carry out a test of the hypothesis that $\beta = 0$. ***(5 marks)***

End of Question Paper

Table of the pth quantile of the χ^2 distribution with ν degrees of freedom, $\chi^2_{p,\nu}$

		ν								
		1	2	3	4	5	6	7	8	9
	0.10	0.016	0.211	0.584	1.064	1.610	2.204	2.833	3.490	4.168
	0.50	0.455	1.386	2.366	3.357	4.351	5.348	6.346	7.344	8.343
p	0.90	2.706	4.605	6.251	7.779	9.236	10.645	12.017	13.362	14.684
	0.95	3.841	5.991	7.815	9.488	11.070	12.592	14.067	15.507	16.919
	0.99	6.635	9.210	11.345	13.277	15.086	16.812	18.475	20.090	21.666