1 (i) Consider a purely toroidal magnetic field of the form

\[ \mathbf{B} = B_0(r) \hat{\theta}, \]

in a cylindrical coordinate system \((r, \theta, y)\). Here, \(B_0(r)\) decreases with increasing \(r\). Assume that the perfectly conducting plasma is initially at rest and is magnetically dominated by such a field, so that we can ignore gas pressure and force due to gravity. If the fluid’s perturbed velocity vector is \(\mathbf{v} = v_r \hat{r} + v_y \hat{y}\), show that the linearised induction equation for the perturbed magnetic field \(\mathbf{B}_1\) is given by

\[ \frac{\partial \mathbf{B}_1}{\partial t} = \frac{B_0}{r} \frac{\partial v_r}{\partial \theta} \hat{r} + B_0 \Phi \hat{\theta} + \frac{B_0}{r} \frac{\partial v_y}{\partial \theta} \hat{y}, \]

where \(\Phi = -\left( \nabla \cdot \mathbf{v} - \frac{v_r}{r} \right) - \frac{v_r}{B_0(r)} \frac{\partial B_0(r)}{\partial r}. \)  

(7 marks)

Write down the equation for \(\frac{\partial^2 \mathbf{v}_1}{\partial t^2}\), using the MHD momentum equation for such a magnetically dominated plasma in terms of \(\Phi\), using the definition for Alfvén speed, \(v_A = v_A(r) = B_0(r)/\sqrt{\mu_0 \rho_0}\) where \(\rho_0\) is the density and \(\mu_0\) is the permeability.  

(7 marks)

(ii) Using a scalar potential \(\psi(x, y)\), we can describe the magnetic field \(\mathbf{B}\) as

\[ \mathbf{B} = \nabla \psi \times \hat{z}. \]

What partial differential equation in \((x, y)\) coordinates does \(\psi\) satisfy? Find a non-trivial solution for this equation which has oscillatory behaviour in \(x\).  

(11 marks)
(i) Consider the magnetic induction equation in the case where the magnetic diffusivity \( \eta = 0 \).
Use \( \nabla \cdot \mathbf{B} = 0 \) and the continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]

to show that the induction equation may be written as

\[
\frac{\partial}{\partial t} \left( \frac{\mathbf{B}}{\rho} \right) + (\mathbf{v} \cdot \nabla) \frac{\mathbf{B}}{\rho} = \left( \frac{\nabla \times \mathbf{B}}{\mu_0} \right) \times \mathbf{v}.
\]

(8 marks)

(ii) An inviscid, perfectly conducting, incompressible fluid, is permeated by a uniform magnetic field \( \mathbf{B}_0 \). The motion of the fluid is described by the momentum equation

\[
\rho \left[ \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} \right] = -\nabla p + \frac{(\nabla \times \mathbf{B})}{\mu_0} \times \mathbf{B}
\]

and magnetic induction equation

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}).
\]

The fluid is initially at rest and then given a small perturbation. Write down the linearised momentum and induction equations. (7 marks)

(iii) Seeking solutions proportional to \( \exp \left( i(k \cdot \mathbf{x} - \omega t) \right) \) in the above linearised equations, show that

\[
\omega^2 = \frac{(k \cdot \mathbf{B}_0)^2}{\mu_0 \rho_0}
\]

where \( \mu_0 \) is the magnetic permeability and \( \rho_0 \) the fluid density. (10 marks)

3 (i) If a plasma is incompressible and the radius of a magnetic flux tube is decreased by a factor 3, use conservation of mass and flux to determine what happens to its length and field strength? (8 marks)

(ii) The momentum equation can be written as

\[
\frac{D \mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g}.
\]

Suppose that \( \mathbf{u} = 0 \) and \( \mathbf{g} = g \hat{z} \) where \( g \) is a constant. Show that the above equation can be written as an equation in \( z \) alone.
For \( p = K \rho^{1+\frac{1}{n}} \) with both \( K \) and \( n \) constant, find \( \rho \) in terms of \( z \). (7 marks)
(iii) If the magnetic field is given by
\[ \mathbf{B} = -y \mathbf{x} + \mathbf{y}, \]
calculate

(a) \( \mathbf{J} \times \mathbf{B} \)  
\( (2 \text{ marks}) \)

(b) \( (\mathbf{B} \cdot \nabla) \frac{\mathbf{B}}{\mu_0} \)  
\( (2 \text{ marks}) \)

(c) \( -\nabla \left( \frac{B^2}{2\mu_0} \right) \)  
\( (2 \text{ marks}) \)

Sketch the field-lines denoting the directions with arrows and indicate clearly on your sketch the direction of the tension forces on \( y = 0 \).  
\( (4 \text{ marks}) \)

4 (i) State Ohm’s law using electric field \( \mathbf{E} \), fluid velocity \( \mathbf{u} \), magnetic field \( \mathbf{B} \) and current density \( \mathbf{J} \).  
\( (2 \text{ marks}) \)

Using \( \mathbf{u} = U_0(-x, y, 0) \), \( \mathbf{B} = (0, B, 0) \) and zero electric field, show that
\[ B(x) \propto \exp \left[ -(U_0/2\eta)x^2 \right], \]
where \( \eta \) is the magnetic diffusivity.  
\( (5 \text{ marks}) \)

(ii) For a linear force-free magnetic field,
\[ \nabla \times \mathbf{B} = \alpha \mathbf{B}, \]
where \( \alpha \) is some function of position. What is the restriction on \( \alpha \) and why?  
\( (3 \text{ marks}) \)

Consider a poloidal magnetic field of the form
\[ \mathbf{B} = B_r \hat{r} + B_z \hat{z}, \]
such that \( \mathbf{B} \) is independent of azimuth \( \phi \).
Show that such an axisymmetric, force-free, poloidal magnetic field must be current free.  
\( (5 \text{ marks}) \)
(iii) For a rotating object symmetric around a rotation axis, the velocity $v$ in cylindrical coordinates $(r, \phi, z)$

$$v = r \Omega(r, z) \hat{\phi}$$

is independent of $\phi$. Here, $\Omega$ is the angular velocity. Now, consider that the object has axisymmetric poloidal field, frozen into plasma. Show that a steady state is possible only if $\Omega$ is constant along field lines.

(Hint: use $B = \nabla \times \frac{1}{r} \psi(r, z) \hat{\phi}$). (10 marks)
\[ \nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \]

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\[
\nabla \cdot \mathbf{V} = \frac{1}{gh} \left[ \frac{\partial}{\partial u} (ghV_u) + \frac{\partial}{\partial v} (fhV_v) + \frac{\partial}{\partial w} (fgV_w) \right]
\]
\[
\nabla \times \mathbf{V} = \frac{1}{gh} \left[ \frac{\partial}{\partial v} (hV_w) - \frac{\partial}{\partial w} (gV_v) \right] \hat{u} + \frac{1}{fh} \left[ \frac{\partial}{\partial w} (fV_u) - \frac{\partial}{\partial u} (hV_w) \right] \hat{v} + \frac{1}{fg} \left[ \frac{\partial}{\partial u} (gV_v) - \frac{\partial}{\partial v} (fV_u) \right] \hat{w}
\]

vector identity:
\[
\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}
\]

End of Question Paper