Throughout this paper, unless otherwise stated, all vector spaces are either over the field of real numbers, \( \mathbb{R} \), or the field of complex numbers, \( \mathbb{C} \).

1. (i) Let \( c_0 \) be the vector space of sequences \( (a_n) \) in \( \mathbb{C} \) such that \( a_n \to 0 \) as \( n \to \infty \), with pointwise addition and scalar multiplication. Prove that we have a norm on \( c_0 \) defined by the formula

\[
\|(a_n)\| = \sup\{|a_n| \mid n \in \mathbb{N}\}.
\]

(4 marks)

(ii) Say what is meant by the statement that a normed vector space is a Banach space.

(2 marks)

(iii) Is the space \( c_0 \) a Banach space? Prove your answer.

(8 marks)

(iv) Say what is meant by a closed subset of a Banach space.

(2 marks)

(v) Let \( l^\infty \) be the Banach space of bounded sequences, \( (a_n) \), in \( \mathbb{C} \), with norm defined as in part (i). Which of the following are closed subsets of \( l^\infty \)? Justify your answer.

(a) \( c_0 \).

(b) The vector space \( c_{00} \) of all sequences \( (a_n) \) of complex numbers for which there exists \( N \) with \( a_n = 0 \) whenever \( n \geq N \).

(c) For a given \( N \), the vector space \( c_N \) of all sequences \( (a_n) \) of complex numbers such that \( a_n = 0 \) whenever \( n \geq N \).

(9 marks)
2 (i) Let $V$ and $W$ be normed vector spaces. Let $T: V \to W$ be a linear map.

(a) Define what is meant by the statement that $T$ is a bounded linear map, and in this case define the norm of $T$. (2 marks)

(b) Define what is meant by the statement that the map $T$ is open. State the open mapping theorem. (3 marks)

(c) Give an example of a surjective bounded linear map between normed vector spaces that is not open. Justify your answer. (4 marks)

(ii) Let $V$ be a normed vector space over the field $\mathbb{R}$.

(a) Define the dual space $V^*$, and prove that it is a Banach space. (9 marks)

(b) State the Hahn-Banach theorem. (2 marks)

(c) Define a linear map $\tau: V \to (V^*)'$ by the formula

$$\tau(v)(f) = f(v) \quad f \in V^*, \ v \in V.$$ 

Use the Hahn-Banach theorem to prove that $\|\tau(v)\| = \|v\|$ for all $v \in V$. (5 marks)
3. (i) State the Stone-Weierstrass theorem for a space of real-valued functions.  
(2 marks)

(ii) Let $A$ be the set of linear combinations of the functions $f_n : [0, \pi] \to \mathbb{R}$ defined by the formula $f_n(x) = \cos(nx)$, where $n$ is a non-negative integer. Use the Stone-Weierstrass theorem to prove that $A$ is dense in $C[0, \pi]$.  
(4 marks)

(iii) Prove that any set which is dense in $C[0, \pi]$ under the supremum norm is also dense in $L^2[0, \pi]$.  
(4 marks)

(iv) Define $e_n \in L^2[0, \pi]$ by the formulae

$$e_0(x) = \frac{1}{\sqrt{\pi}} \quad e_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx), \quad n \geq 1.$$ 

Prove that the set $\{e_0, e_1, e_2, \ldots\}$ is an orthonormal basis for the space $L^2[0, \pi]$.  
(5 marks)

(v) Define a function $f : [0, \pi] \to \mathbb{R}$ by $f(x) = \sin x$. Find coefficients $\alpha_n \in \mathbb{R}$ such that

$$f = \sum_{n=0}^{\infty} \alpha_n e_n$$

and calculate the sum

$$\sum_{n=0}^{\infty} |\alpha_n|^2.$$ 

You may use any standard facts about series involving orthonormal bases of Hilbert spaces without proof.  
(10 marks)
4 (i) (a) Let $A$ be a complex unital Banach algebra, and $x \in A$. Define the spectrum of $x$. (2 marks)

(b) Let $x \in A$ satisfy the inequality $\|x\| < 1$. Prove that $1 - x$ is invertible. (6 marks)

(c) Let $H$ be a complex Hilbert space. Define what is meant by a unitary operator on $H$, and prove that if $U$ is unitary then

$$\text{Spectrum}(U) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}.$$ (6 marks)

(ii) (a) Again, let $A$ be a complex unital Banach algebra, and $x \in A$. Prove that we can define an element $\exp(x)$ by the formula

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and that for all $x, y \in A$ such that $xy = yx$ we have

$$\exp(x + y) = \exp(x) \exp(y).$$ (8 marks)

(b) Let $H$ be a complex Hilbert space, and let $T : H \to H$ be a self-adjoint operator. Show that $\exp(iT)$ is unitary. (3 marks)

5 (i) Define what is meant by the statement that a linear map between normed vector spaces is a compact operator. (2 marks)

(ii) Let $K : V \to W$ be a linear map between normed vector spaces $V$ and $W$. Prove that $K$ is a compact operator if and only if for any bounded sequence $(x_n)$ in $V$, the image $(Kx_n)$ in $W$ has a convergent subsequence. (4 marks)

(iii) Prove that any bounded linear map with finite-dimensional image is compact. (4 marks)

(iv) Let $H$ be a Hilbert space, and $T : H \to H$ be a bounded linear operator. Let $(x_n)$ be a bounded sequence in $H$ such that the sequence $(T^*Tx_n)$ converges. Prove that $(Tx_n)$ is a Cauchy sequence. (7 marks)

(v) Let $K : H \to H$ be a bounded linear operator such that $K^*K$ is compact. Prove that $K$ is also compact. (5 marks)

(vi) Is the converse result to part (v) true? Justify your answer. (3 marks)

End of Question Paper