The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2015–2016

MAS436: Functional Analysis — 2 hours 30 minutes

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

Throughout this paper, unless otherwise stated, all vector spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$

1 (i) Let c_0 be the vector space of sequences (a_n) in $\mathbb{C}$ such that $a_n \to 0$ as $n \to \infty$, with pointwise addition and scalar multiplication. Prove that we have a norm on c_0 defined by the formula

$$\|(a_n)\| = \sup\{|a_n| \mid n \in \mathbb{N}\}.$$

(4 marks)

(ii) Say what is meant by the statement that a normed vector space is a Banach space. ***(2 marks)***

(iii) Is the space c_0 a Banach space? Prove your answer. ***(8 marks)***

(iv) Say what is meant by a closed subset of a Banach space. ***(2 marks)***

(v) Let l^∞ be the Banach space of bounded sequences, (a_n), in $\mathbb{C}$, with norm defined as in part (i). Which of the following are closed subsets of l^∞? Justify your answer.

(a) c_0.

(b) The vector space c_{00} of all sequences (a_n) of complex numbers for which there exists N with $a_n = 0$ whenever $n \geq N$.

(c) For a given N, the vector space c_N of all sequences (a_n) of complex numbers such that $a_n = 0$ whenever $n \geq N$.

(9 marks)

2 (i) Let V and W be normed vector spaces. Let $T: V \to W$ be a linear map.

(a) Define what is meant by the statement that T is a bounded linear map, and in this case define the norm of T. ***(2 marks)***

(b) Define what is meant by the statement that the map T is open. State the open mapping theorem. ***(3 marks)***

(c) Give an example of a surjective bounded linear map between normed vector spaces that is not open. Justify your answer. ***(4 marks)***

(ii) Let V be a normed vector space over the field $\mathbb{R}$.

(a) Define the *dual space* V^*, and prove that it is a Banach space. ***(9 marks)***

(b) State the Hahn-Banach theorem. ***(2 marks)***

(c) Define a linear map $\tau: V \to (V^*)^*$ by the formula

$$\tau(v)(f) = f(v) \qquad f \in V^*,\ v \in V.$$

Use the Hahn-Banach theorem to prove that $\|\tau(v)\| = \|v\|$ for all $v \in V$. ***(5 marks)***

3 (i) State the Stone-Weierstrass theorem for a space of real-valued functions. ***(2 marks)***

(ii) Let A be the set of linear combinations of the functions $f_n\colon [0,\pi] \to \mathbb{R}$ defined by the formula $f_n(x) = \cos(nx)$, where n is a non-negative integer. Use the Stone-Weierstrass theorem to prove that A is dense in $C[0,\pi]$. ***(4 marks)***

(iii) Prove that any set which is dense in $C[0,\pi]$ under the supremum norm is also dense in $L^2[0,\pi]$. ***(4 marks)***

(iv) Define $e_n \in L^2[0,\pi]$ by the formulae

$$e_0(x) = \frac{1}{\sqrt{\pi}} \qquad e_n(x) = \sqrt{\frac{2}{\pi}}\cos(nx), \;\; n \geq 1.$$

Prove that the set $\{e_0, e_1, e_2, \ldots\}$ is an orthonormal basis for the space $L^2[0,\pi]$. ***(5 marks)***

(v) Define a function $f\colon [0,\pi] \to \mathbb{R}$ by $f(x) = \sin x$. Find coefficients $\alpha_n \in \mathbb{R}$ such that

$$f = \sum_{n=0}^{\infty} \alpha_n e_n$$

and calculate the sum

$$\sum_{n=0}^{\infty} |\alpha_n|^2.$$

You may use any standard facts about series involving orthonormal bases of Hilbert spaces without proof.

(10 marks)

4 (i) (a) Let A be a complex unital Banach algebra, and $x \in A$. Define the *spectrum* of x. ***(2 marks)***

(b) Let $x \in A$ satisfy the inequality $\|x\| < 1$. Prove that $1 - x$ is invertible. ***(6 marks)***

(c) Let H be a complex Hilbert space. Define what is meant by a *unitary operator* on H, and prove that if U is unitary then

$$Spectrum(U) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}.$$

(6 marks)

(ii) (a) Again, let A be a complex unital Banach algebra, and $x \in A$. Prove that we can define an element $\exp(x)$ by the formula

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and that for all $x, y \in A$ such that $xy = yx$ we have

$$\exp(x + y) = \exp(x) \exp(y).$$

(8 marks)

(b) Let H be a complex Hilbert space, and let $T\colon H \to H$ be a self-adjoint operator. Show that $\exp(iT)$ is unitary. ***(3 marks)***

5 (i) Define what is meant by the statement that a linear map between normed vector spaces is a *compact operator*. ***(2 marks)***

(ii) Let $K\colon V \to W$ be a linear map between normed vector spaces V and W. Prove that K is a compact operator if and only if for any bounded sequence (x_n) in V, the image (Kx_n) in W has a convergent subsequence. ***(4 marks)***

(iii) Prove that any bounded linear map with finite-dimensional image is compact. ***(4 marks)***

(iv) Let H be a Hilbert space, and $T\colon H \to H$ be a bounded linear operator. Let (x_n) be a bounded sequence in H such that the sequence (T^*Tx_n) converges. Prove that (Tx_n) is a Cauchy sequence. ***(7 marks)***

(v) Let $K\colon H \to H$ be a bounded linear operator such that K^*K is compact. Prove that K is also compact. ***(5 marks)***

(vi) Is the converse result to part (v) true? Justify your answer. ***(3 marks)***

End of Question Paper