SCHOOL OF MATHEMATICS AND STATISTICS  
Spring Semester 2015–2016
Optics and Symplectic Geometry  
2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.
Throughout the paper I denotes an identity matrix and J denotes a matrix of the form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. The standard symplectic form $\Omega$ on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$. 

Turn Over
(i) Let $S$ be a $2n \times 2n$ matrix (with real entries). Define what it means for $S$ to be a symplectic matrix.

Now write $S$ in block form as

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A, B, C, D$ are $n \times n$ matrices. Show that $S$ is symplectic if and only if all the following hold

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I.$$  

(4 marks)

(ii) Suppose that $S \in Sp(2n)$ has a real eigenvalue $\lambda$ with a corresponding eigenvector $Z \in \mathbb{R}^{2n}$. Show that $JZ$ is an eigenvector for $S^T$ and find the corresponding eigenvalue.  

(4 marks)

(iii) Let $(V, \omega)$ be a symplectic vector space of dimension $2n$. Define the concept of symplectic basis for $(V, \omega)$.

Now consider the symplectic vector space $(\mathbb{R}^{2n}, \Omega)$. Let $e_1, \ldots, e_n$ be any orthonormal basis in $\mathbb{R}^n$. Prove that

$$(e_1, 0), \ldots, (e_n, 0), (0, e_1), \ldots, (0, e_n)$$

is a symplectic basis in $\mathbb{R}^{2n}$.  

(8 marks)

(iv) Take $X \in \mathbb{R}^3$ and $Y \in \mathbb{R}^3$ with $|X| = 1$ and $X \cdot Y = 0$. You are given that $V := \{(\xi, \eta) \in \mathbb{R}^6 \mid X \cdot \xi = 0, \quad X \cdot \eta + \xi \cdot Y = 0\}$.

is a symplectic subspace of $(\mathbb{R}^6, \Omega)$.

(a) Let $X, B_2, B_3$ be an orthonormal basis for $\mathbb{R}^3$ containing $X$, and write $Y$ with respect to this basis as $Y = y_2 B_2 + y_3 B_3$.

In terms of coordinates with respect to this basis, write down the conditions that $(\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3$ belong to $V$.

For $(\xi, \eta), (\xi', \eta') \in V$, express $\Omega((\xi, \eta), (\xi', \eta'))$ in terms of these coordinates.

Find a symplectic basis for $V$ in terms of $X, B_2, B_3$.

(b) Now assume that $Y = 0$. Show that the subspace

$$L = \{(\xi, \eta) \in V \mid \xi = \eta\}$$

is Lagrangian in $V$ and find another Lagrangian subspace $L'$ of $V$ such that $V = L \oplus L'$, proving the stated properties.  

(16 marks)
Let \( W \) be a vector space of dimension \( k \).
Define the dual space \( W^* \), being sure to include definitions of the vector space operations. State, without proof, the dimension of \( W^* \). (6 marks)

Let \((V, \omega)\) be a symplectic vector space of dimension \( 2n \) and let \( L_1 \) and \( L_2 \) be Lagrangian subspaces such that \( V = L_1 \oplus L_2 \).
Define a map \( \Phi: L_1 \rightarrow L_2^* \) by \((\Phi(v_1))(v_2) = \omega(v_1, v_2)\) for \( v_1 \in L_1 \) and \( v_2 \in L_2 \). State why \( \Phi \) takes values in \( L_2^* \) and show that it is a linear map.
Prove that \( \Phi \) is an isomorphism of vector spaces, stating clearly any general result of linear algebra which you use. (12 marks)

Snell’s Law may be given as \( n' \sin \theta' = n \sin \theta \) for refraction across a boundary between mediums with indexes of refraction \( n \) and \( n' \), where \( \theta \) and \( \theta' \) are the angles made by the rays with the normal to the boundary.
Derive Snell’s Law in vector form so that it applies to rays in \( \mathbb{R}^3 \), including a simple diagram in your answer. Your answer should be in terms of unit vectors \( v \) and \( v' \) along the incoming and outgoing rays, a unit vector \( \Sigma \) normal to the boundary between the two regions, and the indexes of refraction \( n \) and \( n' \). (10 marks)

Calculate the matrix product
\[
\begin{bmatrix}
I & w' I \\
0 & I
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
M & I
\end{bmatrix}
\begin{bmatrix}
I & w I \\
0 & I
\end{bmatrix}
\]
where \( M \) is symmetric, and describe the optical situation which leads to finding the product of these matrices. (12 marks)

Let \( S \in Sp(4) \) be given in block form by \([A B; C D]\). Suppose that there are real numbers \( w, w' > 0 \) such that
\[
A - I = w'C, \quad D - I = wC.
\]
Assuming that \( C \) is invertible, express \( B \) in terms of \( w, w' \) and \( C \). (5 marks)
In this question each $\mathbb{R}^{2n}$ has the standard symplectic form $\Omega$.

(a) Let $W$ be an $n$–dimensional subspace of $\mathbb{R}^{2n}$. Take a basis of $W$ and write the elements as the columns of a $2n \times n$ matrix which we write in block form as

$$\begin{bmatrix} M \\ N \end{bmatrix}$$

where $M$ and $N$ are $n \times n$ matrices.

Prove that $W$ is Lagrangian if and only if $M^T N$ is symmetric. (4 marks)

(b) Let $L \subseteq \mathbb{R}^{2n}$ be a Lagrangian subspace of $\mathbb{R}^{2n}$. Show that it is transverse to both $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$ if and only if it has a representation as in (a) of the form $\begin{bmatrix} M \\ I \end{bmatrix}$ with $M$ invertible. (8 marks)

(c) Let $L$ and $L'$ be Lagrangian subspaces which are both transversal to $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$. State without proof the theorem which gives criteria for the existence of $S \in Sp(2n)$ such that $S(\mathbb{R}^n \times 0) = \mathbb{R}^n \times 0$, $S(0 \times \mathbb{R}^n) = 0 \times \mathbb{R}^n$, and $S(L) = L'$. (3 marks)

(d) Let $L$ be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$(1, 3, 5, 1, 0, 0), \quad (0, 1, 0, 1, -2, 1), \quad (0, 1, 2, 5, 0, -1).$$

Show that $L$ is Lagrangian in $(\mathbb{R}^6, \Omega)$ and that it is transverse to both $\mathbb{R}^3 \times 0$ and $0 \times \mathbb{R}^3$. Represent $L$ in the form $\begin{bmatrix} M \\ I \end{bmatrix}$ as in (b) and determine the signature of $M$. (8 marks)

End of Question Paper