1  (i)  (a) What does it mean to say that a topological space is compact? What does it mean to say that a topological space is Hausdorff? (4 marks)

(b) Prove that any continuous bijection from a compact space to a Hausdorff space is a homeomorphism. (6 marks)

(c) Show that the quotient space $\mathbb{B}^2/S^1$ (the closed disc mod the boundary circle) is homeomorphic to $S^2$. (3 marks)

(ii) (a) What is a path from $a$ to $b$ in a topological space $X$? (2 marks)

(b) Define the fundamental group $\pi_1(X,x_0)$. [You should define the underlying set and the group multiplication, and check the multiplication is well defined. You need not check the group axioms are satisfied.] (6 marks)

(c) Show that for any two topological spaces $X, Y$, with basepoints $x_0 \in X, y_0 \in Y$, the fundamental group $\pi_1(X \times Y; (x_0, y_0))$ is isomorphic to the product of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$. (4 marks)
(i) (a) What is a covering map? (4 marks)

(b) State the Path Lifting Lemma for a covering map $p : Y \rightarrow X$, and explain how it can be used to define a function 

$$\ell : \pi_1(X, x_0) \rightarrow p^{-1}(x_0),$$

where $x_0 \in X$. State conditions under which $\ell$ is a bijection. (8 marks)

(ii) Let 

$$SU(2) = \{ A \in M_2(\mathbb{C}) \mid AA^T = I, \det(A) = 1 \}$$

be the $2 \times 2$ special unitary group.

(a) Show that 

$$SU(2) = \{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \}.$$

Use this to show that $\pi_1(SU(2), I)$ is the trivial group, where $I$ is the identity matrix. (8 marks)

(b) Consider the subgroup $Z = \{ I, -I \}$ of $SU(2)$ and let $PSU(2) = SU(2)/Z$. Determine the fundamental group of $PSU(2)$ using the image of $I$ as basepoint. (5 marks)

3 (i) (a) What is a chain complex of abelian groups? What does it mean to say that two chain maps $\theta, \phi : C_* \rightarrow D_*$ are chain homotopic? (5 marks)

(b) Show that if $\theta$ and $\phi$ are chain homotopic, they induce the same map in homology. (4 marks)

(ii) (a) Show that if $K$ is a simplicial complex, $P$ is a new vertex and $c_PK$ is the $P$-cone on $K$ then $H_i(c_PK) = 0$ for $i > 0$. Explain the relevance of this to the homology of the standard abstract simplex $\Delta^n$. (8 marks)

(b) If $K$ consists of all subsets of $\{0, 1, 2, 3, 4\}$ with $\leq 3$ elements (i.e., the 2-skeleton of $\Delta^4$), calculate $H_*(K)$. (8 marks)
4. (i) (a) Suppose that $C_\bullet$ is a chain complex with only finitely many terms non-zero and all terms finite dimensional vector spaces. What is the Lefschetz number $\Lambda(\theta)$ of a chain map $\theta : C_\bullet \rightarrow C_\bullet$? (2 marks)

(b) Show that $\Lambda(\theta) = \Lambda(\theta_\ast)$ where $\theta_\ast : H_\ast(C_\bullet) \rightarrow H_\ast(C_\bullet)$ is the induced map in homology. (6 marks)

(c) State the Lefschetz Fixed Point Theorem. (2 marks)

(ii) Consider maps $f : T \rightarrow T$, where $T$ denotes the 2-torus.

(a) Calculate the homology of $T$. (6 marks)

(b) Is there a map $f : T \rightarrow T$ homotopic to the identity which has no fixed points? Justify your answer. (2 marks)

(c) Suppose that $f$ induces the identity map on $H_0(T)$ and $H_2(T)$ and that $f_\ast(x) = -x$ for $x \in H_1(T)$. Calculate the Lefschetz number of $f$ and deduce that $f$ must have a fixed point. Describe such a map $f$ which has exactly 4 fixed points. (3 marks)

(d) Suppose that a group $G$ of order 2 acts on $T$ with 4 fixed points, and that $T/G$ is an orientable surface. Find the genus of $T/G$. (4 marks)

5. Are the following true or false. Justify your answers.

(i) Any self-map of a contractible space has a fixed point. (5 marks)

(ii) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$. (5 marks)

(iii) The Klein bottle admits the structure of a topological group. (5 marks)

(iv) The Euler characteristic distinguishes the homotopy types of connected one dimensional simplicial complexes. (5 marks)

(v) If $K$ and $L$ are simplicial complexes, any continuous function $f : |K| \rightarrow |L|$ is homotopic to a map $|s|$ for a simplicial map $s : K \rightarrow L$. (5 marks)

End of Question Paper