The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2015–16

Introduction to Relativity — **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) (a) State the two postulates of special relativity. ***(4 marks)***

(b) Define *inertial frame*. Give two examples of frames that are approximately inertial. ***(4 marks)***

(ii) The inertial frame $\tilde{R} : (c\tilde{t}, \tilde{x})$ is moving at a constant velocity v relative to the inertial frame $R : (ct, x)$, such that the two frames are related by the two-dimensional *Lorentz transformation*

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \qquad \gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

(a) Draw a labelled sketch of $\gamma(v)$ over the domain $v \in (-c, c)$. ***(4 marks)***

(b) Consider a particle moving with uniform velocity u in R, such that its coordinates are $x = ut$. Apply the Lorentz transformation to find the velocity $\tilde{u} = \tilde{x}/\tilde{t}$ of the particle in $\tilde{R}$. ***(4 marks)***

(c) An inertial observer O sees two rockets A and B fly past in opposite directions with uniform velocities $-c/2$ and $c/2$, respectively. Find the velocity of rocket B as seen from the inertial frame of rocket A. ***(4 marks)***

(d) Briefly describe the phenomenon of *time dilation* in special relativity. According to A, how long does it take for 1 minute to pass on a clock on board B? ***(5 marks)***

2 A Lorentz transformation L is represented by a 4×4 matrix satisfying

$$L^T g L = g, \qquad \text{where} \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

with L^T denoting the matrix transpose.

(i) Show that:

(a) $g^2 = I$, where I is the identity matrix. ***(2 marks)***

(b) $(\det L)^2 = 1$. ***(3 marks)***

(c) $(L^{-1})^T g L^{-1} = g$,
i.e. the inverse L^{-1} is also a Lorentz transformation. ***(3 marks)***

(d) $L^T = g L^{-1} g$. ***(2 marks)***

(e) the transpose L^T is also a Lorentz transformation. ***(2 marks)***

(ii) Let

$$L = \begin{pmatrix} 2 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

Show that L is a *proper orthochronous* Lorentz transformation.
(13 marks)

3 Consider a particle A and a ray of light B moving in an inertial frame R, with four-velocities

$$U = \gamma(u)\begin{pmatrix} c \\ u \\ 0 \\ 0 \end{pmatrix}, \qquad V = c\begin{pmatrix} 1 \\ \cos\theta \\ \sin\theta \\ 0 \end{pmatrix},$$

respectively, where u and θ are constants: $0 < u < c, \quad 0 \leq \theta < \pi$.
Let $\tilde{R}$ be the rest frame of particle A, in which $\tilde{U} = (c, 0, 0, 0)^T$.

(i) (a) Define the Lorentz bracket.
Show that $g(U, U) = c^2$ and $g(V, V) = 0$.
Classify U and V as timelike, spacelike or null, respectively.
(5 marks)

(b) Write down the standard Lorentz transformation L between the inertial frames R and $\tilde{R}$, such that $\tilde{U} = LU = (c, 0, 0, 0)^T$. ***(3 marks)***

(c) Find the components of the four-velocity $\tilde{V}$ of the light ray in $\tilde{R}$.
(3 marks)

(d) In frame R, the light ray makes an angle θ with the x-axis.
The angle θ may be found by taking a ratio of components of the four-velocity: $\cos\theta = V^1/V^0$.
By taking a ratio of components of $\tilde{V}$, show that the light ray makes an angle $\tilde{\theta}$ with the $\tilde{x}$-axis in $\tilde{R}$ given by

$$\cos\tilde{\theta} = \frac{\cos\theta - \frac{u}{c}}{1 - \frac{u}{c}\cos\theta}.$$

(3 marks)

(ii) An astronomer at the origin of R observes three distant stars X, Y, Z, at angles $\theta = 0$, $\theta = \pi/6$ and $\theta = \pi/2$, respectively. She concludes that the three stars are identical: each star appears as a small circle with the identical intensity, colour and angular width.

Now consider an astronomer at the origin of $\tilde{R}$ with $u = \dfrac{\sqrt{3}\,c}{2}$.

(a) Find the angles $\tilde{\theta}$ for the three stars X, Y, Z, using part (i)(d).
Sketch the light rays from X, Y, Z in the $(\tilde{x}, \tilde{y})$ plane. ***(5 marks)***

(b) Show that $\dfrac{d\theta}{d\tilde{\theta}} = \gamma(u)\left(1 - \dfrac{u}{c}\cos\theta\right)$.
[*Hint:* You may first find $\sin\tilde{\theta}$ using the method of part (i)(d)].
Comment on the appearance of the three stars in $\tilde{R}$. ***(6 marks)***

4 A particle in an inertial frame R moves on a circular trajectory of radius r with displacement four-vector

$$X(\tau) = \begin{pmatrix} c\tau \cosh\rho \\ r\cos(\omega\tau) \\ r\sin(\omega\tau) \\ 0 \end{pmatrix},$$

where ρ, r and ω are positive constants, and τ is proper time.

(i) Sketch the *world-line* of the particle on a 2D spacetime diagram, with vertical axis ct and horizontal axis x. ***(4 marks)***

(ii) (a) Find the four-velocity $V(\tau) = \dfrac{dX}{d\tau}$.
Use $g(V, V) = c^2$ to deduce that

$$\sinh\rho = \omega r/c.$$

(5 marks)

(b) Find the four-acceleration $A(\tau)$.
Show that $g(V, A) = 0$ and $g(A, A) = -a^2$, where $a = \omega^2 r$.
(6 marks)

(iii) An advanced satellite travels on a circular orbit around a small asteroid, by using a thruster to generate an acceleration of $a = 10^8\text{ms}^{-2}$ in its instantaneous rest frame.
According to the satellite's clock, its orbital period is 1 minute.
In the asteroid's inertial rest frame R, neglecting the effects of gravity, calculate:

(a) the radius of the orbit; ***(4 marks)***

(b) the speed of the rocket; ***(3 marks)***

(c) the orbital period. ***(3 marks)***

5 (i) (a) Define the *rest mass* and *four-momentum* of a particle. ***(2 marks)***

(b) A particle of rest mass $m > 0$ has three-velocity $\mathbf{v}$ in an inertial frame R, in which its energy is

$$E = \gamma(v)mc^2, \qquad v = |\mathbf{v}|.$$

Write down an expression for the three-momentum $\mathbf{p}$ in R. ***(2 marks)***

(c) Show that $E^2 - p^2c^2 = m^2c^4$. ***(3 marks)***

(d) Expand E as a Maclaurin series in v, up to and including the term at order v^4.
Give some physical interpretation of the first two terms in this series. ***(4 marks)***

(e) The Sun converts its rest mass into energy using nuclear fusion, generating a power of approximately $3.96 \times 10^{26} Js^{-1}$. How many years would it take the Sun to convert a mass equal to the mass of the Earth (5.972×10^{24}kg)? ***(4 marks)***

(ii) A particle of rest mass M is in uniform motion along the x-axis of inertial frame R with speed u. It splits into two particles each of rest mass m, such that the first particle is stationary in R and the second has uniform speed $v = \sqrt{3}\, c/2$. Use conservation of four-momentum to show that

$$u = c/\sqrt{3}, \qquad M = \sqrt{6}\, m.$$ ***(10 marks)***

End of Question Paper