1 (i) Verify that the d’Alembert general solution

\[ u(x, t) = f(x - ct) + g(x + ct), \]

where \( f \) and \( g \) are arbitrary functions and \( c \) is a constant, satisfies the wave equation

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \]  \hspace{1cm} (1)

Hence find the transverse displacement, for time \( t > 0 \), of an infinite stretched string that, at \( t = 0 \), is at rest with displacement \( \sin(x) \), where \( x \) is the distance along the string.

\( \text{(10 marks)} \)

(ii) The transverse displacement \( u \) of a stretched string, held fixed at its end-points \( x = 0 \) and \( x = L \), satisfies the wave equation given by (1). At \( t = 0 \) the displacement is zero. Verify that

\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin(\alpha_n x) \sin(\alpha_n ct) \]

satisfies equation (1) and the initial and boundary conditions, where \( B_n \) and \( \alpha_n \) (\( n = 1, 2, 3, \ldots \)) are constants to be determined.

If it is further given that, at \( t = 0 \), the velocity \( \partial u/\partial t = f(x) \), find an integral formula for \( B_n \).

\( \text{(15 marks)} \)

2 A string of mass per unit length \( \rho \) is under a tension \( \rho c^2 \), where \( \rho \) and \( c \) are constants. Its equilibrium position is \( 0 \leq x \leq L, y = 0 \). The string undergoes transverse vibrations and its displacement is \( y(x, t) \), where

\[ \frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}, \]

and \( y(0, t) = y(L, t) = 0 \).

(i) Verify that all these conditions are satisfied by

\[ y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left\{ a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right\}, \]

where \( \{a_n\}, \{b_n\} \) are constants. [\textit{Note that you are asked to verify this result, not derive it.}]

\( \text{(7 marks)} \)
2 (continued)

(ii) Find \(\{a_n\}\) and \(\{b_n\}\) for the case when
\[
y(x, 0) = 0, \quad y(x, 0) = (V/l^3)x(l/2 - x)(l - x).
\]

(13 marks)

(iii) Show that the initial kinetic energy of the string is
\[
\frac{\rho V^2 l}{1680}
\]

(5 marks)

3 (A model of a stethoscope.) Sound waves propagate in the positive Oz direction inside the circular cylinder \(r = a\) (where \(r^2 = x^2 + y^2\) in standard notation). The velocity potential \(\phi\) satisfies
\[
c^2 \left\{ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right\} = \frac{\partial^2 \phi}{\partial t^2},
\]
where the constant \(c\) is the speed of sound.

(i) State how \(c\) depends on pressure \((p)\) and density \((\rho)\). Determine the value of \(c\) for the case when this is
\[
\left( \frac{p}{p_0} \right) = \left( \frac{\rho}{\rho_0} \right)^\gamma,
\]
where \(\gamma = 1.4\), and \(p_0\) and \(\rho_0\) are the ambient pressure and density with \(p_0 \approx 1.013 \times 10^5 \text{ N m}^{-2}, \rho_0 \approx 1.293 \text{ kg m}^{-3}\).

(6 marks)

(ii) Seek solutions of (2) of the form
\[
\phi = g(r) \exp\{i(kz - \omega t)\},
\]
where \(k\) and \(\omega\) are real positive constants. Show that
\[
g''(r) + \frac{1}{r} g'(r) + m^2 g(r) = 0
\]
where \(m^2\) is a constant, depending on \(\omega, c\) and \(k\). (You may assume that \(m^2 > 0\).)

(7 marks)
3 (continued)

(iii) It is given that $\phi$ is bounded at $r = 0$, that $\frac{\partial \phi}{\partial r} = 0$ at $r = a$, and that the only solution of (3) that is bounded at $r = 0$ must be a multiple of $J_0(mr)$, where $J_0(\xi)$ is the Bessel function of order zero. Show that $m = m_n (n = 1, 2, \ldots)$, where $m_n = \beta_n/a$ and $\beta_n$ is the $n$th non-zero root of $J_0'(\xi) = 0$. Given that the $\beta_n$ are discrete, that $\beta_1 < \beta_2 < \ldots$, and that $\beta_n \to \infty$ as $n \to \infty$, deduce that, for fixed $\omega$, there are a finite number of positive values of $k$.

(12 marks)

4 The equilibrium position of the free surface of a liquid of infinite depth is $z = 0$, where $z$ is measured vertically upwards. A surface wave causes the displacement of this surface to be $\eta(x, t)$, where $x$ is measured along the undisturbed surface and

$$\eta = a \sin(kx - \omega t),$$

with $a$, $k$ and $\omega$ being positive constants with $a$ small.

The velocity potential is $\phi(x, z, t)$ and satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

You are given that: (a) $\frac{\partial \phi}{\partial z} \to 0$ as $z \to -\infty$; (b) $\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}$ at $z = 0$;

(c) $\frac{\partial \phi}{\partial t} + g\eta = 0$ at $z = 0$.

(i) Explain briefly the physical meaning of each of (a), (b), and (c).

(5 marks)

(ii) Find $\phi(x, z, t)$ and show that the dispersion relation is

$$\omega^2 = gk.$$

(14 marks)

(iii) Determine the phase velocity $c$ and the group velocity $c_g$ in terms of $k$. State two quantities that are propagated with speed $c_g$.

(6 marks)
In a model of traffic flow in the direction of $Ox$, the density of traffic at time $t$ is $\rho(x, t)$, and it is assumed that the velocity of traffic of density $\rho$ is $v = v(\rho)$.

(i) Show that

$$\rho_t + c(\rho)\rho_x = 0,$$

where $c(\rho) = d(\rho v)/d\rho$.

(4 marks)

(ii) Given that $\rho = f(x)$ at $t = 0$ for $-\infty < x < \infty$, and that

$$c(f(\xi)) = F(\xi),$$

show that, for $t \geq 0$, $\rho = f(\xi)$ on the curve $x = \xi + F(\xi)t$.

(7 marks)

(iii) Show that the above solution breaks down on any curve for which $F'(\xi) < 0$.

(2 marks)

(iv) In a particular case

$$v(\rho) = \frac{V}{P}(P - \rho) \quad (0 \leq \rho \leq P),$$

where $V$ and $P$ are constant. Given that

$$\rho(x, 0) = \begin{cases} 0 & (x \leq 0), \\ \rho_R(x^2/L^2) & (0 \leq x \leq L), \\ \rho_R & (x \geq L), \end{cases}$$

where $L$ and $\rho_R$ are constants with $\rho_R < P$, obtain the solution to the traffic flow equation in (i). Determine when and where the solution first breaks down.

(12 marks)

End of Question Paper