The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2015–16

MAS331 Metric Spaces — 2 hours 30 minutes

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) (a) Define the *supremum metric* d_∞ on the set $C[0,1]$ of continuous functions on $[0,1]$ and prove that it is a metric.

(b) Show that if $f_n \to f$ in $(C[0,1], d_\infty)$ then $f_n(x) \to f(x)$ for any $x \in [0,1]$.

(14 marks)

(ii) (a) Let (X, d) be a metric space. What is a *closed subset* of (X, d)?

(b) Let $a, b \colon [0,1] \to \mathbb{R}$ be functions and define

$$D_{a,b} = \{f \in C[0,1] \mid a(x) \leqslant f(x) \leqslant b(x) \text{ for all } x \in [0,1]\}.$$

Use (i)(b) to show that $D_{a,b}$ is a closed subset of $(C[0,1], d_\infty)$.

(6 marks)

(iii) Let (f_n) be a sequence in $(C[0,1], d_\infty)$ with the property that

$$-x^2 \leqslant f_n(x) \leqslant x^2 \quad \text{for all } x \in [0,1]$$

and suppose that $f_n \to f$. Show that

$$-\frac{1}{3} \leqslant \int_0^1 f(x)dx \leqslant \frac{1}{3}.$$

(5 marks)

2 (i) Let (x_n) be a sequence in a metric space (X, d).

(a) Show that (x_n) has at most one limit.

(b) Let (x_{n_k}) be a subsequence of (x_n). Show that if $x_n \to x$ then $x_{n_k} \to x$ also.

(10 marks)

(ii) Now let $\mathbb{R}$ be equipped with its usual metric and consider the sequence (x_n) defined by

$$x_n = \cos\left(\frac{n\pi}{2} + \frac{\pi}{2n}\right).$$

(a) Show that the subsequence (x_{4k}) converges to 1.

(b) Show that the subsequence (x_{4k+2}) converges to -1.

(c) Use parts (i)(b), (ii)(a) and (ii)(b) to show that (x_n) does *not* have a limit.

(11 marks)

(iii) Prove that *no* subsequence of the sequence (x_n) defined by $x_n = n$ converges to any limit.

(4 marks)

3 Throughout this question $\mathbb{R}^2$ is given the usual Euclidean metric d_2.

(i) (a) Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ be a sequence in $(\mathbb{R}^2, d_2)$. Show that $(x_n, y_n) \to (x, y)$ if and only if $x_n \to x$ and $y_n \to y$.

(b) Use part (i)(a) to show, in terms of ϵ and N, that the sequence $\left(\frac{n-1}{n+1}, \frac{1}{n}\right)$ converges to (1,0).

(15 marks)

(ii) (a) Define, in terms of convergence of sequences, what it means for a function $f\colon (X, d_X) \to (Y, d_Y)$ to be *continuous*.

(b) Let $f\colon \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) = (f_1(x), f_2(x))$$

for functions $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$. Use Part (i)(a) to show that f is continuous if and only if f_1 and f_2 are continuous.

(10 marks)

4 (i) (a) What does it mean for a sequence (x_n) in a metric space (X, d) to be *Cauchy*?

(b) What does it mean for (X, d) to be *complete*?

(c) State without explanation which of the two metric spaces $(C[0,1], d_\infty)$, $(C[0,1], d_1)$ are complete.

(6 marks)

(ii) Let (f_n) be the sequence in $(C[0,1], d_\infty)$ defined by

$$f_n(x) = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \cdots + \frac{x^n}{2^n}.$$

(a) Show that $d_\infty(f_n, f_m) = \dfrac{1}{2^n}\left(1 - \dfrac{1}{2^{m-n}}\right)$ for $m \geq n$.

(b) Show that (f_n) is a Cauchy sequence.

(c) Indicate briefly why the sequence (f_n) converges.

(10 marks)

(iii) Let (g_n) be the sequence in $(C[0,1], d_1)$ defined by

$$g_n(x) = \begin{cases} 0 & 0 \leqslant x \leqslant \frac{1}{2} - \frac{1}{2n}, \\ 2n\left(x - \dfrac{1}{2} + \dfrac{1}{2n}\right) & \frac{1}{2} - \frac{1}{2n} \leqslant x \leqslant \frac{1}{2}, \\ 1 & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

(a) Sketch the graph of g_n, labelling the main features.

(b) Show that $d_1(g_n, g_m) = \dfrac{1}{4n} - \dfrac{1}{4m}$ for $m \geq n$.

(c) Show that (g_n) is a Cauchy sequence.

(9 marks)

5 (i) (a) What is a *contraction* of a metric space (X, d)?

(b) State and prove the *Contraction Mapping Principle*. ***(18 marks)***

(ii) Consider the function $f\colon [1, \infty) \to [1, \infty)$ defined by $f(x) = x + \dfrac{1}{x}$.

(a) Show that $|f(x) - f(y)| < |x - y|$ for all $x, y \in [1, \infty)$.

(b) Show that f does not have a fixed point.

(c) Indicate briefly why this does not contradict the contraction mapping principle. ***(7 marks)***

End of Question Paper