Answer **four** questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

1 (i) (a) Define the *supremum metric* $d_\infty$ on the set $C[0, 1]$ of continuous functions on $[0, 1]$ and prove that it is a metric.

(b) Show that if $f_n \to f$ in $(C[0, 1], d_\infty)$ then $f_n(x) \to f(x)$ for any $x \in [0, 1]$.

(14 marks)

(ii) (a) Let $(X, d)$ be a metric space. What is a *closed subset* of $(X, d)$?

(b) Let $a, b : [0, 1] \to \mathbb{R}$ be functions and define

$$D_{a,b} = \{f \in C[0, 1] \mid a(x) \leq f(x) \leq b(x) \text{ for all } x \in [0, 1]\}.$$  

Use (i)(b) to show that $D_{a,b}$ is a closed subset of $(C[0, 1], d_\infty)$.

(6 marks)

(iii) Let $(f_n)$ be a sequence in $(C[0, 1], d_\infty)$ with the property that

$$-x^2 \leq f_n(x) \leq x^2 \quad \text{for all } x \in [0, 1]$$

and suppose that $f_n \to f$. Show that

$$-\frac{1}{3} \leq \int_0^1 f(x) \, dx \leq \frac{1}{3}.$$  

(5 marks)

[Turn Over]
2

(i) Let \((x_n)\) be a sequence in a metric space \((X, d)\).

(a) Show that \((x_n)\) has at most one limit.

(b) Let \((x_{n_k})\) be a subsequence of \((x_n)\). Show that if \(x_n \to x\) then \(x_{n_k} \to x\) also.

\((10 \text{ marks})\)

(ii) Now let \(\mathbb{R}\) be equipped with its usual metric and consider the sequence \((x_n)\) defined by

\[ x_n = \cos \left( \frac{n\pi}{2} + \frac{\pi}{2n} \right). \]

(a) Show that the subsequence \((x_{4k})\) converges to 1.

(b) Show that the subsequence \((x_{4k+2})\) converges to −1.

(c) Use parts (i)(b), (ii)(a) and (ii)(b) to show that \((x_n)\) does not have a limit.

\((11 \text{ marks})\)

(iii) Prove that no subsequence of the sequence \((x_n)\) defined by \(x_n = n\) converges to any limit.

\((4 \text{ marks})\)

3

Throughout this question \(\mathbb{R}^2\) is given the usual Euclidean metric \(d_2\).

(i) (a) Let \((x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\) be a sequence in \((\mathbb{R}^2, d_2)\). Show that \((x_n, y_n) \to (x, y)\) if and only if \(x_n \to x\) and \(y_n \to y\).

(b) Use part (i)(a) to show, in terms of \(\epsilon\) and \(N\), that the sequence \(\left( \frac{n - 1}{n + 1}, \frac{1}{n} \right)\) converges to \((1, 0)\).

\((15 \text{ marks})\)

(ii) (a) Define, in terms of convergence of sequences, what it means for a function \(f: (X, d_X) \to (Y, d_Y)\) to be \emph{continuous}.

(b) Let \(f: \mathbb{R} \to \mathbb{R}^2\) be defined by

\[ f(x) = (f_1(x), f_2(x)) \]

for functions \(f_1, f_2: \mathbb{R} \to \mathbb{R}\). Use Part (i)(a) to show that \(f\) is continuous if and only if \(f_1\) and \(f_2\) are continuous.

\((10 \text{ marks})\)
4  (i)  (a) What does it mean for a sequence \((x_n)\) in a metric space \((X, d)\) to be Cauchy?
(b) What does it mean for \((X, d)\) to be complete?
(c) State without explanation which of the two metric spaces \((C[0, 1], d_{\infty}), (C[0, 1], d_1)\) are complete.

\((6 \text{ marks})\)

(ii) Let \((f_n)\) be the sequence in \((C[0, 1], d_{\infty})\) defined by
\[
f_n(x) = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \cdots + \frac{x^n}{2^n}.
\]
(a) Show that \(d_{\infty}(f_n, f_m) = \frac{1}{2^n} \left( 1 - \frac{1 - \frac{1}{2^{m-n}}}{2^{m-n}} \right)\) for \(m \geq n\).
(b) Show that \((f_n)\) is a Cauchy sequence.
(c) Indicate briefly why the sequence \((f_n)\) converges.

\((10 \text{ marks})\)

(iii) Let \((g_n)\) be the sequence in \((C[0, 1], d_1)\) defined by
\[
g_n(x) = \begin{cases} 
0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{2^n}, \\
2n \left( x - \frac{1}{2} + \frac{1}{2n} \right) & \frac{1}{2} - \frac{1}{2n} \leq x \leq \frac{1}{2}, \\
1 & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]
(a) Sketch the graph of \(g_n\), labelling the main features.
(b) Show that \(d_1(g_n, g_m) = \frac{1}{4n} - \frac{1}{4m}\) for \(m \geq n\).
(c) Show that \((g_n)\) is a Cauchy sequence.

\((9 \text{ marks})\)

5  (i)  (a) What is a contraction of a metric space \((X, d)\)?
(b) State and prove the Contraction Mapping Principle. \((18 \text{ marks})\)

(ii) Consider the function \(f : [1, \infty) \to [1, \infty)\) defined by \(f(x) = x + \frac{1}{x}\).
(a) Show that \(|f(x) - f(y)| < |x - y|\) for all \(x, y \in [1, \infty)\).
(b) Show that \(f\) does not have a fixed point.
(c) Indicate briefly why this does not contradict the contraction mapping principle.

\((7 \text{ marks})\)

End of Question Paper