Let \( n \geq 1 \). Use the Binomial Theorem to show that
\[
\frac{1 - (1 - x)^n}{x} = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} x^{i-1}.
\]

\((3\text{ marks})\)

(b) Show that
\[
\frac{1 - (1 - x)^n}{x} = 1 + y + \cdots + y^{n-1},
\]
where \( y = 1 - x \).

\((2\text{ marks})\)

(c) By integrating the expression in part (a), show that
\[
\sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i} \binom{n}{i}.
\]

\((6\text{ marks})\)

(ii) (a) How many solutions are there of the equation
\[
x_1 + x_2 + \cdots + x_k = n,
\]
in which each \( x_i \) is a non-negative integer? Give a brief reason for your answer.

\((3\text{ marks})\)

(b) State the Inclusion/Exclusion Principle.

\((3\text{ marks})\)

(c) How many solutions are there of the equation
\[
x_1 + x_2 + x_3 + x_4 = 21,
\]
in which each \( x_i \) is a non-negative integer, such that \( x_1 < 7 \), \( x_2 < 9 \)
and \( x_3 < 12 \) ?

\((8\text{ marks})\)
The numbers 1 to 10 are written in a row. Can one insert plus and minus signs between them in such a way that the value of the resulting expression is zero? \([3 \text{ marks}]\)

Consider a \(3 \times n\) rectangle with the three squares in one corner removed. (The case \(n = 6\) is pictured below.)

Show that this cannot be completely covered by non-overlapping dominoes (that is, by pieces which cover exactly two adjacent squares). \([5 \text{ marks}]\)

(iii) (a) State the Pigeon-hole Principle. \([2 \text{ marks}]\)

(b) Show that in any group of five people, there are two who have the same number of friends within the group. \([5 \text{ marks}]\)

(c) Show that there exists an integer whose decimal representation consists entirely of 1s (that is, an integer of the form 111...1) which is divisible by 1789. \([5 \text{ marks}]\)

(iv) (a) Find distinct representatives of the sets

\[
A_1 = \{1, 2, 7\},
A_2 = \{5, 6, 8\},
A_3 = \{1, 3, 7\},
A_4 = \{2, 3, 4, 7\},
A_5 = \{1, 2, 6, 8\}.
\]

\([1 \text{ mark}]\)

(b) Can distinct representatives of these sets be chosen to include 5, 6 and 8? \([2 \text{ marks}]\)

(c) State a necessary and sufficient condition for sets \(A_1, A_2, \ldots, A_n\) to have distinct representatives. \([2 \text{ marks}]\)
3

(i) Calculate the rook polynomial of the (unshaded) board $B$:

(ii) Let $B$ be part of an $n \times n$ board with rook polynomial

\[ 1 + r_1 x + r_2 x^2 + \cdots + r_n x^n \]

and let $\overline{B}$ be the complement of $B$. Prove that the number of ways of placing $n$ non-challenging rooks on $\overline{B}$ is

\[ \sum_{k=0}^{n} (-1)^k(n - k)!r_k, \]

where $r_0 = 1$.

(iii) (a) Calculate the coefficient of $x^5$ in the rook polynomial of $\overline{B}$, where $B$ is the board in part (i).

(b) Using a relationship between permutations and non-challenging rooks, or otherwise, find the number of permutations of $\{1, 2, 3, 4, 5\}$ satisfying the following conditions.

\[ 1 \mapsto 2, 2 \not\mapsto 4, 2 \not\mapsto 5, 3 \not\mapsto 1, 3 \not\mapsto 2, 4 \not\mapsto 1, 4 \not\mapsto 2. \]
4 (i) For what value of $x$ can the following Latin rectangle be extended to a $6 \times 6$ Latin square?

$$
\begin{pmatrix}
3 & 1 & 2 & 4 \\
1 & 3 & 6 & 2 \\
4 & 6 & x & 3 \\
\end{pmatrix}
$$

Write down one such extension. \hfill (7 marks)

(ii) (a) Show that there is a tournament of $n$ players with scores

$$(n - 1, n - 2, n - 3, \ldots, 2, 1, 0).$$

(b) Hence show that there is a tournament of $3n$ players with scores

$$(2n-1, 2n-1, 2n-1, 2n-2, 2n-2, 2n-2, \ldots, n+1, n+1, n+1, n, n, n).$$

\hfill (4 marks)

(iii) Consider a $4 \times 4$ board with squares labelled by the numbers $1, 2, \ldots, 16$ as shown.

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
$$

Define blocks as follows. For each square on the board, form a block consisting of the number on that square together with the nine numbers not sharing a row or column with that square. For example, \{1, 6, 7, 8, 10, 11, 12, 14, 15, 16\} is a block, corresponding to the top left square.

(a) Show that each number is in 10 blocks. \hfill (2 marks)

(b) Show that each pair of numbers appears in precisely 6 blocks. \hfill (6 marks)

(c) Deduce that the blocks make up a $(16, 16, 10, 10, 6)$ design. \hfill (3 marks)

End of Question Paper