SCHOOL OF MATHEMATICS AND STATISTICS

Computational Engineering Mathematics

*Marks will be awarded for your best FOUR answers*
The figure shows a rectangular plate made of a homogeneous isotropic material. The plate is divided into intervals of equal length 20 cm in the $x$ and $y$ directions. The temperature $T(x, y)$ in this plate satisfies the indicated boundary conditions (given in °C) and has reached a steady-state condition so that it is described by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$ 

(a) Draw a sketch of the solution domain, showing clearly the line of symmetry for $T(x, y)$, and indicating which of the unknown temperatures $T_1, \ldots, T_6$ are equal to each other. (5 marks)

(b) Use the finite difference formulae on the formula sheet to find the finite difference equations required to find estimates of the nodal temperatures $T_1, T_2$ and $T_3$. (10 marks)

(c) Express the finite difference equations in part (b) in the form $AT = b$, where $A$ is a $3 \times 3$ matrix, $T = (T_1, T_2, T_3)^T$ and $b = (100, 30, 40)^T$, with units in °C, where you should give the matrix $A$. Hence, using Gaussian elimination or otherwise, show that

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 310 \\ 230 \\ 170 \end{bmatrix} \text{ °C}.$$ (10 marks)
The temperature $T(x, t)$ satisfies the convection-diffusion equation

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} \quad (0 \leq x \leq 1). \quad (1)$$

(a) If $T_{ij} = T(x_i, t_j)$, with $i = 0$ and $i = N$ corresponding to $x = 0$ and $x = 1$, respectively, and $j = 0$ corresponding to $t = 0$, use backward differences for time derivatives and central differences for space derivatives to derive the implicit scheme

$$-(k - \beta)T_{i+1,j} + (1 + 2k)T_{ij} - (k + \beta)T_{i-1,j} = T_{ij-1}$$

for $i = 1, \ldots, N - 1$ and $j = 1, 2, \ldots$, where

$$k = \frac{\Delta t}{(\Delta x)^2} \quad \text{and} \quad \beta = \frac{\Delta t}{2\Delta x}. \quad (5 \text{ marks})$$

(b) Equation (1) is to be solved (approximately) over $0 \leq x \leq 1$, with boundary conditions $T(0, t) = 20$ and $T(1, t) = 30$, and initial temperature distribution $T(x, 0) = 20 + 10x$.

Taking $\Delta x = 0.25$ and $\Delta t = 0.1$, use the implicit scheme in part (a) to write down the system of equations for the temperature at $x = 0.25, 0.5, 0.75$ and time $t = 0.1$. (Note that you do not need to solve the equations.)

(12 marks)

(c) Show from your answer to part (b) that the Jacobi iteration equations to find the $(k + 1)$th iteration from the $k$th iteration are

$$\begin{bmatrix} T_{1,1}^{(k+1)} \\ T_{2,1}^{(k+1)} \\ T_{3,1}^{(k+1)} \end{bmatrix} = \frac{1}{4.2} \begin{bmatrix} 58.5 \\ 25 \\ 69.5 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 3 & 7 & 0 \\ 0 & 3 & 7 \end{bmatrix} \begin{bmatrix} T_{1,1}^{(k)} \\ T_{2,1}^{(k)} \\ T_{3,1}^{(k)} \end{bmatrix} \quad (8 \text{ marks})$$
You are given that
\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \]
where \( \sigma_{ij} \) is the stress tensor, \( \varepsilon_{ij} \) is the (symmetric) strain tensor, and \( C_{ijkl} \) is a fourth order tensor of constant coefficients. In the following, \( \delta_{ij} \) is the Kronecker delta tensor.

(a) Given that
\[ C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \]
for constants \( \lambda \) and \( \mu \), show that
\[ \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{mm} + 2\mu \varepsilon_{ij}. \]  
(5 marks)

Show that the mean normal stress \( \frac{1}{3} \sigma_{kk} \) is
\[ \frac{1}{3} \sigma_{kk} = \left( \lambda + \frac{2}{3} \mu \right) \varepsilon_{mm}. \]  
(3 marks)

(b) Given that \( \lambda = 0.6391 \text{ Pa} \) and \( \mu = 0.9068 \text{ Pa} \), and that the strain tensor satisfies \( \varepsilon_{11} = -0.3014 \times 10^{-2} \), \( \varepsilon_{22} = 0.5315 \times 10^{-2} \), \( \varepsilon_{33} = 0.4353 \times 10^{-2} \), \( \varepsilon_{12} = 0.3124 \times 10^{-2} \), \( \varepsilon_{23} = 0.7518 \times 10^{-2} \), and \( \varepsilon_{31} = -0.5416 \times 10^{-2} \),
find the stress tensor.

Hence find the stress force on a unit area element in the direction \( n = (0.5774, -0.5774, 0.5774)^T \).  
(6 marks)
4 The velocity field in a fluid is given by \( \mathbf{v} = u \mathbf{i} + v \mathbf{j} + w \mathbf{k} \), and the density by \( \rho \), where \( u, v, w \) and \( \rho \) are functions of \( x, y, z \) and \( t \).

(a) The divergence of \( \mathbf{v} \) is defined by

\[
\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{1}{\Delta V} \frac{D}{Dt}(\Delta V),
\]

where \( \Delta V \) is an infinitesimal control volume and \( \frac{D}{Dt} \) is the substantial derivative. By considering the mass of the moving control volume, \( \Delta m = \rho \Delta V \), derive the equation of continuity (i.e. of mass conservation), and show that it can be written as

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{8 marks}
\]

(b) Let \( x = x_1 \), \( y = x_2 \) and \( z = x_3 \).

(i) The \( i \) component of the curl of a vector \( \mathbf{u} \), \( \nabla \times \mathbf{u} \), can be written using index notation as

\[
(\nabla \times \mathbf{u})_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k,
\]

where \( \varepsilon_{ijk} \) is the Levi-Civita tensor.

Show that

\[
\varepsilon_{ijk} \frac{\partial}{\partial x_j} \left( \frac{\partial \phi}{\partial x_k} \right) = 0,
\]

i.e.

\[
\nabla \times \nabla \phi = 0,
\]

for any \( \phi \). \tag{4 marks}

(ii) The vorticity \( \mathbf{\omega} \) is defined by

\[
\mathbf{\omega} = \nabla \times \mathbf{v}, \quad \text{i.e. } \omega_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k.
\]

Show that

\[
(\mathbf{v} \times \mathbf{\omega})_i = \frac{\partial}{\partial x_i} \left( \frac{1}{2} v_j v_j \right) - v_j \frac{\partial}{\partial x_j} v_i. \tag{4 marks}
\]

[ You may assume that \( \varepsilon_{kij} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \). ]

(iii) The momentum equation may be written as

\[
\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{1}{\rho} F_i,
\]

where \( p \) is the pressure, \( \sigma_{ij} \) is the stress tensor, and \( \mathbf{F} \) is the body force.
Show that the momentum equation can be rewritten as
\[
\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \frac{1}{2} v_j v_j \right) - (v \times \omega)_i = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} (\sigma_{ij}) + \frac{1}{\rho} F_i, \tag{2}
\]
and show that
\[
[\nabla \times (v \times \omega)]_i = v_i \frac{\partial \omega_j}{\partial x_j} + v_j \frac{\partial \omega_i}{\partial x_j} - \frac{\partial}{\partial x_j} (v_j \frac{\partial v_i}{\partial x_j}). \tag{5 \text{ marks}}
\]
[Again you may assume that \(\varepsilon_{kij} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}.\)]

Hence, by taking the curl of (2), show that
\[
\frac{\partial \omega_i}{\partial t} + v_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial v_i}{\partial x_j}
\]
if the density \(\rho\) is constant, the body force is conservative (i.e. \(\nabla \times F = 0\)) and the flow is incompressible (i.e. \(\nabla \cdot v = 0\)).

(4 marks)

[You may assume that \(\nabla \cdot (\nabla \times u) = 0\) for any \(u\).]
A one-dimensional bar has variable cross-sectional area $A(x)$, is fixed at one end ($x = 0$) and is loaded axially by a known force $F_t$ at its free end ($x = \ell$). The total force acting across the cross-sectional area located at $x$ is denoted by $F(x)$. The body force per unit length is $f(x)$.

(a) The strong form of the one-dimensional equation of force balance is

$$\frac{d}{dx} \left( AE \frac{du}{dx} \right) + f = 0,$$

where $u(x)$ is the displacement in the $x$ direction (so the strain $\varepsilon_{xx} = du/dx$) and the stress $\sigma_{xx} = E\varepsilon_{xx}$, where $E$ is a constant. The boundary conditions are $u(0) = 0$ and $F(\ell) = F_t$.

By multiplying this by an arbitrary weighting function $w(x) > 0$, derive the weak form of the equation:

$$\int_0^\ell \frac{dw}{dx} AE \frac{du}{dx} dx = w(\ell)F(\ell) - w(0)F(0) + \int_0^\ell w f \, dx \quad \text{(7 marks)}$$

(b) Consider a solution domain $0 \leq x \leq \ell$, with two nodes (at $x_1 = 0$ and $x_2 = \ell$). The trial solution is $u(x) \approx U_1 N_1^1(x) + U_2 N_2^1(x) = \mathbf{N}^T \mathbf{U}$ where $\mathbf{N} = (N_1^2, N_2^2)^T$, $\mathbf{U} = (U_1, U_2)^T$, and $U_1$ and $U_2$ are constants. The weight function is $w(x) = c_1 N_1^2(x) + c_2 N_2^2(x) = c^T \mathbf{N}$, where $c = (c_1, c_2)^T$ and $c_1$ and $c_2$ are arbitrary constants. $N_1^1$ and $N_2^1$ are defined by

$$N_1^1(x) = \frac{x - x_1}{x_2 - x_1}, \quad N_2^1(x) = \frac{x - x_1}{x_2 - x_1}.$$

(i) Show that

$$E \left[ \int_0^\ell A \frac{d\mathbf{N}}{dx} \frac{d\mathbf{N}^T}{dx} \right] \mathbf{U} = \mathbf{N}(\ell)F(\ell) - \mathbf{N}(0)F(0) + \int_0^\ell \mathbf{N} f \, dx. \quad \text{(4 marks)}$$

(ii) Given that $A$ and $f$ are constant, obtain the pair of simultaneous finite element equations

$$U_1 - U_2 = \frac{\ell}{AE} \left( \frac{\ell f}{2} - F(0) \right), \quad U_2 - U_1 = \frac{\ell}{AE} \left( \frac{\ell f}{2} + F(\ell) \right). \quad \text{(9 marks)}$$

Given that $F(0) = \ell f + F(\ell)$, deduce that

$$u(x) \approx \frac{x}{AE} \left( \frac{\ell f}{2} + F(\ell) \right). \quad \text{(5 marks)}$$

End of Question Paper
Formula Sheet

Notation:

\[ U(x_i, t_j) \equiv U_{i,j} \]

Forward difference formula for \( \partial U / \partial t \):

\[
\frac{\partial U}{\partial t}(x_i, t_j) \approx \frac{U_{i,j+1} - U_{i,j}}{\Delta t}
\]

Forward difference formula for \( \partial U / \partial x \):

\[
\frac{\partial U}{\partial x}(x_i, t_j) \approx \frac{U_{i+1,j} - U_{i,j}}{\Delta x}
\]

Backward difference formula for \( \partial U / \partial t \):

\[
\frac{\partial U}{\partial t}(x_i, t_j) \approx \frac{U_{i,j} - U_{i,j-1}}{\Delta t}
\]

Backward difference formula for \( \partial U / \partial x \):

\[
\frac{\partial U}{\partial x}(x_i, t_j) \approx \frac{U_{i,j} - U_{i-1,j}}{\Delta x}
\]

Central difference formula for \( \partial U / \partial x \):

\[
\frac{\partial U}{\partial x}(x_i, t_j) \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}
\]

Central difference formula for \( \partial^2 U / \partial x^2 \):

\[
\frac{\partial^2 U}{\partial x^2}(x_i, t_j) \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2}
\]
Relation between different parameters:

A number of relationships between $E$, $\nu$, $K$, $\lambda$ and $\mu$ hold and are summarized in Table 1. $\mu$ ($\equiv G$) is the elastic shear modulus, $K$ the elastic bulk modulus, $E$ the elastic stiffness (or Young’s Modulus) and $\nu$ Poisson’s ratio.

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$\nu$</th>
<th>$K$</th>
<th>$\lambda$</th>
<th>$\mu \equiv G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E, \nu$</td>
<td>-</td>
<td>-</td>
<td>$E$</td>
<td>$\frac{E\nu}{(1 + \nu)(1 - 2\nu)}$</td>
<td>$\frac{E}{2(1 + \nu)}$</td>
</tr>
<tr>
<td>$E, K$</td>
<td>-</td>
<td>$\frac{3K - E}{6K}$</td>
<td>-</td>
<td>$\frac{K(9K - 3E)}{9K - E}$</td>
<td>$\frac{3KE}{9K - E}$</td>
</tr>
<tr>
<td>$K, \mu$</td>
<td>$\frac{9\mu K}{3K + \mu}$</td>
<td>$\frac{3K - 2\mu}{2(3K + \mu)}$</td>
<td>-</td>
<td>$K - \frac{2\mu}{3}$</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: The relations between the properties of elastic bodies.