Answer five questions. If you answer more than five questions, only your best five will be counted.

1. Consider a system with $N$ degrees of freedom. The Lagrange–function is given by $L = L(q_i, \dot{q}_i, t)$, with $i = 1, \ldots, N$.

   (i) Write down the Euler–Lagrange equation corresponding to the coordinate $q_i$. Furthermore, by explicit calculation, show that

   $$ \frac{d}{dt} \left( \sum_{i=1}^{N} q_i \frac{\partial L}{\partial \dot{q}_i} - L \right) = -\frac{\partial L}{\partial t}. $$

   Hence, show that if $L$ does not explicitly depend on time, the quantity in the brackets is conserved. What is the interpretation of the quantity? (7 marks)

   (ii) The Lagrange–function for a particle of mass $m$ and charge $e$ is given by

   $$ L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e\Phi + e (\dot{x}A_x + \dot{y}A_y + \dot{z}A_z), $$

   where $x, y, z$ are the coordinates of the particle, $\Phi$ is the scalar potential, $A_x, A_y, A_z$ are the components of the vector potential $A = (A_x, A_y, A_z)$ and $\dot{x} = dx/dt$, etc. Show that the Euler–Lagrange equations lead to the equation

   $$ m\ddot{x} = e (E + v \times B), $$

   where $x = (x, y, z)$ is the position vector of the particle, $v = dx/dt$ is the velocity, $E = -\nabla \Phi - \frac{\partial A}{\partial t}$ and $B = \nabla \times A$. Hint: Since $x, y,$ and $z$ appear in $L$ on the same footing, it is sufficient to consider one component only (e.g. the $x$–coordinate). (13 marks)
Consider a system with $N$ degrees of freedom, which is specified by $N$ generalised coordinates $q_i$ $(i = 1, \ldots, N)$ and a Lagrangian $L = L(q_i, \dot{q}_i, t)$.

(i) State Hamilton’s principle. (2 marks)

(ii) Define the canonical momenta $P_i$ and define the Hamiltonian $H = H(q_i, P_i, t)$. (3 marks)

(iii) Consider the action

$$S = \int_{t_1}^{t_2} \left[ \sum_{i=1}^{N} P_i \dot{q}_i - H \right] dt.$$ 

Show that extremising this action with respect to $q_i$ and $P_i$ leads to Hamilton’s equations. Hint: You may want to use the Euler–Lagrange equations for $q_i$ and $P_i$. (8 marks)

(iv) A particle moves along the $x$-axis under the influence of the potential $V = \frac{1}{2} k x^2 + \lambda x^4$. Write down the Hamilton–function $H(x, P)$ for this system but do not assume that $H$ is the sum of the kinetic and potential energies. Find the equation of motion for the particle from Hamilton’s equations. Is $H$ conserved? (7 marks)

3 (i) Consider two functions $f = f(q_i, P_i)$ and $g = g(q_i, P_i)$, where $i = 1, \ldots, N$ and $N$ is the number of degrees of freedom. Define the Poisson bracket $\{f, g\}$.

(2 marks)

For the rest of this question, we consider a system which is described by one degree of freedom. The generalised coordinate is denoted by $q$ and the canonical momentum by $P$.

(ii) Consider a coordinate transformation $q \to Q = Q(q, P), P \to \tilde{P} = \tilde{P}(q, P)$ and denote the Poisson brackets in the two coordinate systems with

$$\{f, g\}_{q, P} \equiv \frac{\partial f}{\partial q} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial q},$$

$$\{f, g\}_{Q, \tilde{P}} \equiv \frac{\partial f}{\partial Q} \frac{\partial g}{\partial \tilde{P}} - \frac{\partial f}{\partial \tilde{P}} \frac{\partial g}{\partial Q}.$$ 

Show that $\{f, g\}_{q, P} = \{f, g\}_{Q, \tilde{P}}$ only if $\{Q, \tilde{P}\}_{q, P} = 1$. Hint: It is useful to first show that $\{f, g\}_{q, P} = \{f, g\}_{Q, \tilde{P}} \cdot \{Q, \tilde{P}\}_{q, P}$. (13 marks)

(iii) Show that $\{Q, \tilde{P}\}_{q, P} = 1$ for the following coordinate transformation:

(5 marks)

$$q \to Q = \log \left( \frac{\sinh q}{P} \right), \quad P \to \tilde{P} = P \frac{\cosh q}{\sinh q}.$$
The electromagnetic field strength tensor is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A_\mu$ is a four–vector.

(i) How does this tensor transform under general Lorentz transformations? (3 marks)

(ii) In general, $F_{\mu\nu}$ can be written in terms of the electric field $\mathbf{E} = (E_x, E_y, E_z)$ and magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ as follows

\[
F_{\mu\nu} = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -B_z & B_y \\
-E_y & B_z & 0 & -B_x \\
-E_z & -B_y & B_x & 0
\end{pmatrix}.
\]

Assume that in an inertial frame, called frame $R_1$, $F_{\mu\nu}$ has the form above. In this system, the electric field is non–zero, i.e. $\mathbf{E} = (E_x, E_y, E_z)$ is not vanishing but the magnetic field vanishes, i.e. $\mathbf{B} = (B_x, B_y, B_z) = (0, 0, 0)$. Consider now a second inertial frame, called $R_2$, moving with velocity $\mathbf{v} = (v, 0, 0)$ in the $x$–direction relative to frame $R_1$. The Lorentz–transformation between frames $R_1$ and $R_2$ is given by the matrix

\[
\Lambda^\mu_\nu = \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

with $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$. Assume that in frame $R_2$ the electromagnetic field strength tensor has the same form as in frame $R_1$, i.e. it can be written as

\[
F'_{\mu\nu} = \begin{pmatrix}
0 & E'_x & E'_y & E'_z \\
-E'_x & 0 & -B'_z & B'_y \\
-E'_y & B'_z & 0 & -B'_x \\
-E'_z & -B'_y & B'_x & 0
\end{pmatrix}.
\]

Determine the magnetic field $\mathbf{B}'$ in the system $R_2$ and show that it doesn’t vanish (i.e. $\mathbf{B}' = (B'_x, B'_y, B'_z) \neq 0$). (17 marks)
For this question, you can use the expression for $F_{\mu\nu}$ given in Question 4.

(i) Show that $F_{\mu\nu}F^{\mu\nu} = 2\left(B^2 - E^2\right)$, where $E = (E_x, E_y, E_z)$ is the electric field and $B = (B_x, B_y, B_z)$ is the magnetic field. Is this an invariant under a Lorentz–transformation? (8 marks)

(ii) Consider the following Lagrangian, describing the interaction between the electromagnetic field and a real massless scalar field $\phi$:

$$L = \frac{1}{2} \eta^{\mu\nu} \left( \partial_\mu \phi \right) \left( \partial_\nu \bar{\phi} \right) - e^{-\phi} F_{\mu\nu} F^{\mu\nu}.$$ 

Use the Euler–Lagrange equation for $\phi$ to show that the equation of motion for the scalar field is given by $\partial_\mu \partial^\mu \phi - 2e^{-\phi}(B^2 - E^2) = 0$. Add a mass term to the Lagrangian above. What is the equation of motion for the scalar field then? (12 marks)

(i) By setting $\phi = \phi_1 + i\phi_2$, show that the two Lagrangians $L_s$ and $L_c$ are equivalent. (7 marks)

(ii) Use the Euler–Lagrange equations for $\phi_1$ and $\phi_2$ to find the equations of motion for the two fields $\phi_1$ and $\phi_2$. Using the equations of motion for $\phi_1$ and $\phi_2$, find the equation of motion for $\phi = \phi_1 + i\phi_2$. (5 marks)

(iii) Show that the action $S = \int L_c d^4x$ is invariant under the transformation

$$\phi \rightarrow \phi' = e^{i\alpha} \phi, \quad x^\mu \rightarrow x'^\mu = x^\mu$$

where $\alpha$ is a real constant. Find the (conserved) Noether current. You are given that the Noether current can be written as

$$j^\mu = -\frac{\partial L_c}{\partial (\partial_\mu \phi)} F_1(\phi, \partial \phi) - \frac{\partial L_c}{\partial (\partial_\mu \bar{\phi})} F_2(\phi, \partial \bar{\phi}),$$

for general infinitesimal transformations of the form ($\epsilon$ is a small parameter)

$$x'^\mu = x^\mu$$

$$\phi'(x') = \phi(x) + \epsilon F_1(\phi, \partial \phi)$$

$$\bar{\phi}'(x') = \bar{\phi}(x) + \epsilon F_2(\phi, \partial \phi)$$

Hint: Assume that $\alpha$ is small and expand the transformations (3) to bring them into the form (4). (8 marks)

End of Question Paper