Data provided: Formula Sheet

MAS420

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015–16**

MAS420 Signal Processing **2 hours**

Attempt all FOUR questions.

1 (i) You are given that the set $\phi_n(t) = e^{in\sigma t} : -\infty < n < \infty$, where $\sigma = \dfrac{2\pi}{T}$, forms an orthonormal basis for the Hilbert space of finite power signals of period T with inner product

$$(f, g) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)g^*(t)\ dt.$$

Prove Parseval's theorem

$$\|f\|^2 = \sum_{n=-\infty}^{n=\infty} |c_n|^2,$$

where $c_n = (f, \phi_n)$. ***(4 marks)***

(ii) Find the complex Fourier coefficients for the periodic signal $f(t)$, where $f(t) = t$ for $-\pi \leq t < \pi$ and $f(t + 2\pi) = f(t)$. ***(7 marks)***

(iii) Use Parseval's theorem and your result in (ii) to derive the equality

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(5 marks)

(iv) The signal is transmitted over a link which will not pass frequencies greater than $\dfrac{\pi}{2}$ rad s^{-1}, but passes other frequencies unchanged. The received signal is $g(t)$. Find an expression for $g(t)$ as a sine/cosine series and calculate the percentage of the power that is lost during transmission. ***(9 marks)***

2 (i) Prove the convolution theorem

$$f * g(t) \leftrightarrow F(\omega)G(\omega),$$

where $f(t)$ and $g(t)$ are signals with Fourier transforms $F(\omega)$ and $G(\omega)$ respectively. ***(4 marks)***

(ii) Use the convolution theorem to evaluate

$$g(t) = \mathrm{sinc}(\Omega(t - t_0)) * \mathrm{sinc}(\Psi(t - t_1))$$

for $\Psi > \Omega$. ***(6 marks)***

(iii) (a) Define the following:

- a linear shift invariant (LSI) system;
- the system transfer function (STF), without any reference to the Fourier transform or the impulse response function;
- the impulse response function (IRF), without reference to the STF or convolution.

(4 marks)

(b) A system, S, uses integration to smooth noisy signals, i.e. if the input signal is $k(t)$, the output is given by

$$g_{out}(t) = \int_{t-T}^{t} k(s)\, ds,$$

where T is a constant.

(α) You are **given** that S is linear and shift-invariant. Verify that its STF is given by

$$H(\omega) = Te^{-iT\omega/2}\,\mathrm{sinc}\left(\frac{T\omega}{2}\right).$$

(3 marks)

(β) Find the impulse response function, $h(t)$, that corresponds to S. ***(2 marks)***

(γ) Use the STF to find the output from the system if the input is

$$k(t) = 1 + 4\sin\frac{\pi t}{T} + \cos\frac{2\pi t}{T},$$

simplifying your answer as much as possible. ***(6 marks)***

3 (i) If $f(t)$ has a real, non-negative Fourier transform, $F(\omega)$, prove that $f(0) \geq 0$ and $|f(t)| \leq f(0)$ for any t. ***(4 marks)***

(ii) Define the equivalent rectangle resolution, τ, of a signal, $f(t)$, stating clearly the conditions under which it is defined. Explain, using a clear diagram, how it is related to the signal $f(t)$. ***(5 marks)***

(iii) Consider the signal $f(t) = 5\,\mathrm{sinc}^2(6t)$. Calculate and sketch the signal spectrum $F(\omega)$. ***(2 marks)***

For the signal $f(t)$, find

(a) its bandwidth, Ω (rad/s), ***(1 mark)***

(b) its energy, and ***(4 marks)***

(c) the equivalent rectangle resolution, τ.

Verify that for this signal $\tau\Omega > \pi$. ***(4 marks)***

The signal is passed through a low-pass filter with system transfer function $H(\omega) = p_6(\omega)$ to produce the signal $g(t)$. Calculate the ratio of the energy of $g(t)$ to the original signal, $f(t)$. ***(5 marks)***

4 (i) Assuming the Fourier transform pair $\overline{\delta}_T(t) \leftrightarrow \sigma\overline{\delta}_\sigma(\omega)$, where $\overline{\delta}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, $\overline{\delta}_\sigma(\omega)$ is defined similarly and $\sigma = 2\pi/T$, prove that

$$f_s(t) \equiv f(t)\overline{\delta}_T(t) \leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\sigma),$$

where $F(\omega)$ is the Fourier transform of $f(t)$. ***(5 marks)***

(ii) Using the previous result, show that if $f(t)$ is Ω-bandlimited and $T < \pi/\Omega$, then $f(t)$ can be recovered exactly from the sampled signal $f_s(t)$ by the sinc interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\mathrm{sinc}\left\{\frac{\sigma}{2}(t - nT)\right\}.$$

Clear diagrams are likely to help your answer. ***(6 marks)***

(iii) Find the Nyquist frequency, in Hz, of the signal

$$f(t) = \mathrm{sinc}^2(8t).$$ ***(4 marks)***

(iv) This signal is sampled at 3/4 of the Nyquist frequency and the samples are used to form a signal $g(t)$ by sinc interpolation. Making use of clear diagrams, find $G(\omega)$ and hence $g(t)$. ***(10 marks)***

End of Question Paper

Formula sheet

Function Definitions:

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs:

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a}p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a}q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a+i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t-t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega-\omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \dfrac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t-t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega-\omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\mathrm{e}^{-in\sigma t}\ \mathrm{d}t$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\cos n\sigma t\ \mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\sin n\sigma t\ \mathrm{d}t$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\ \mathrm{d}\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\ \mathrm{d}s \longleftrightarrow F(\omega)G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$