SCHOOL OF MATHEMATICS AND STATISTICS

MAS420 Signal Processing

Attempt all FOUR questions.

1  (i) You are given that the set $\phi_n(t) = e^{i\sigma t} : -\infty < n < \infty$, where $\sigma = \frac{2\pi}{T}$, forms an orthonormal basis for the Hilbert space of finite power signals of period $T$ with inner product

$$ (f, g) = \frac{1}{T} \int_{-T/2}^{T/2} f(t)g^*(t) \, dt. $$

Prove Parseval's theorem

$$ \|f\|^2 = \sum_{n=-\infty}^{n=\infty} |c_n|^2, $$

where $c_n = (f, \phi_n)$.  

(4 marks)

(ii) Find the complex Fourier coefficients for the periodic signal $f(t)$, where $f(t) = t$ for $-\pi \leq t < \pi$ and $f(t + 2\pi) = f(t)$.  

(7 marks)

(iii) Use Parseval's theorem and your result in (ii) to derive the equality

$$ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. $$

(5 marks)

(iv) The signal is transmitted over a link which will not pass frequencies greater than $\frac{\pi}{2}$ rad s$^{-1}$, but passes other frequencies unchanged. The received signal is $g(t)$. Find an expression for $g(t)$ as a sine/cosine series and calculate the percentage of the power that is lost during transmission. 

(9 marks)
Prove the convolution theorem

\[ f \ast g(t) \leftrightarrow F(\omega)G(\omega), \]

where \( f(t) \) and \( g(t) \) are signals with Fourier transforms \( F(\omega) \) and \( G(\omega) \) respectively. \( \text{(4 marks)} \)

Use the convolution theorem to evaluate

\[ g(t) = \text{sinc}(\Omega(t - t_0)) \ast \text{sinc}(\Psi(t - t_1)) \]

for \( \Psi > \Omega \). \( \text{(6 marks)} \)

(iii) (a) Define the following:

- a linear shift invariant (LSI) system;
- the system transfer function (STF), without any reference to the Fourier transform or the impulse response function;
- the impulse response function (IRF), without reference to the STF or convolution.

\( \text{(4 marks)} \)

(b) A system, \( S \), uses integration to smooth noisy signals, i.e. if the input signal is \( k(t) \), the output is given by

\[ g_{out}(t) = \int_{t-T}^{t} k(s) \, ds, \]

where \( T \) is a constant.

(\( \alpha \)) You are given that \( S \) is linear and shift-invariant. Verify that its STF is given by

\[ H(\omega) = Te^{-iT\omega/2} \text{sinc} \left( \frac{T\omega}{2} \right). \]

\( \text{(3 marks)} \)

(\( \beta \)) Find the impulse response function, \( h(t) \), that corresponds to \( S \). \( \text{(2 marks)} \)

(\( \gamma \)) Use the STF to find the output from the system if the input is

\[ k(t) = 1 + 4\sin \frac{\pi t}{T} + \cos \frac{2\pi t}{T}, \]

simplifying your answer as much as possible. \( \text{(6 marks)} \)
3 (i) If \( f(t) \) has a real, non-negative Fourier transform, \( F(\omega) \), prove that \( f(0) \geq 0 \) and \( |f(t)| \leq f(0) \) for any \( t \). 

(4 marks)

(ii) Define the equivalent rectangle resolution, \( \tau \), of a signal, \( f(t) \), stating clearly the conditions under which it is defined. Explain, using a clear diagram, how it is related to the signal \( f(t) \). 

(5 marks)

(iii) Consider the signal \( f(t) = 5\text{sinc}^2(6t) \). Calculate and sketch the signal spectrum \( F(\omega) \). 

For the signal \( f(t) \), find 
(a) its bandwidth, \( \Omega \) (rad/s), 
(b) its energy, and 
(c) the equivalent rectangle resolution, \( \tau \). 

Verify that for this signal \( \tau \Omega > \pi \). 

(4 marks)

The signal is passed through a low-pass filter with system transfer function \( H(\omega) = \rho_0(\omega) \) to produce the signal \( g(t) \). Calculate the ratio of the energy of \( g(t) \) to the original signal, \( f(t) \). 

(5 marks)

4 (i) Assuming the Fourier transform pair \( \delta_T(t) \leftrightarrow \sigma \delta_\sigma(\omega) \), where \( \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \), \( \delta_\sigma(\omega) \) is defined similarly and \( \sigma = 2\pi/T \), prove that 

\[
 f_S(t) = f(t) \delta_T(t) \leftrightarrow \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\sigma),
\]

where \( F(\omega) \) is the Fourier transform of \( f(t) \). 

(5 marks)

(ii) Using the previous result, show that if \( f(t) \) is \( \Omega \)-bandlimited and \( T < \pi/\Omega \), then \( f(t) \) can be recovered exactly from the sampled signal \( f_S(t) \) by the sinc interpolation formula

\[
 f(t) = \sum_{n=-\infty}^{\infty} f(nT) \text{sinc} \left\{ \frac{\sigma}{2}(t-nT) \right\}.
\]

Clear diagrams are likely to help your answer. 

(6 marks)

(iii) Find the Nyquist frequency, in Hz, of the signal

\[ f(t) = \text{sinc}^2(8t). \]

(4 marks)

(iv) This signal is sampled at 3/4 of the Nyquist frequency and the samples are used to form a signal \( g(t) \) by sinc interpolation. Making use of clear diagrams, find \( G(\omega) \) and hence \( g(t) \). 

(10 marks)

End of Question Paper
Function Definitions:

Rectangular pulse:
\[ p_a(t) = \begin{cases} 
1 & |t| \leq a \\
0 & |t| > a 
\end{cases} \]

Triangular pulse:
\[ q_a(t) = \begin{cases} 
1 - |t|/a & |t| \leq a \\
0 & |t| > a 
\end{cases} \]

Step function:
\[ U(t) = \begin{cases} 
1 & t \geq 0 \\
0 & t < 0 
\end{cases} \]

Fourier Transform Pairs:
\[
\begin{align*}
p_a(t) & \leftrightarrow 2a \text{sinc}(aw) \\
q_a(t) & \leftrightarrow a \text{sinc}^2(aw/2) \\
\text{sinc}(at) & \leftrightarrow \frac{\pi}{a} p_a(\omega) \\
\text{sinc}^2(at) & \leftrightarrow \frac{\pi}{a} q_a(\omega) \\
e^{-at}U(t) & \leftrightarrow \frac{1}{a+\text{i}\omega} \\
\delta(t) & \leftrightarrow 1 \\
\delta(t-t_0) & \leftrightarrow e^{-\text{i}\omega t_0} \\
1 & \leftrightarrow 2\pi \delta(\omega) \\
e^{\text{i}\omega t} & \leftrightarrow 2\pi \delta(\omega - \omega_0) \\
e^{-t^2/\sigma^2} & \leftrightarrow \sigma \sqrt{2\pi} e^{-\omega^2/2\sigma^2} 
\end{align*}
\]

Fourier transform:
\[ F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\text{i}\omega t} \, dt \]

Inverse Fourier transform:
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{\text{i}\omega t} \, d\omega \]

Duality theorem: If \( f(t) \leftrightarrow F(\omega) \) then \( F(t) \leftrightarrow 2\pi f(-\omega) \)

Scaling: If \( f(t) \leftrightarrow F(\omega) \) then \( f(at) \leftrightarrow \frac{1}{|a|} F(\omega/a) \).

Translation: If \( f(t) \leftrightarrow F(\omega) \) then \( f(t-t_0) \leftrightarrow e^{-\text{i}\omega t_0} F(\omega) \).

Frequency Shift: If \( f(t) \leftrightarrow F(\omega) \) then \( e^{\text{i}\omega_0 t} f(t) \leftrightarrow F(\omega - \omega_0) \).
**Fourier Series:** If $f_T(t)$ is periodic with period $T$ then, with $\sigma = 2\pi / T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{i n \sigma t}$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-i n \sigma t} \, dt$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos n \sigma t + b_n \sin n \sigma t \right)$$

where

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \cos n \sigma t \, dt$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \sin n \sigma t \, dt$$

**Parseval's Theorem:** If $V$ is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for $V$ and $f = \sum_n c_n \phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

**Plancherel's Theorem:** If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t) g^*(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G^*(\omega) \, d\omega$$

**Energy Theorem:** If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

**Convolution Theorem:** If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f * g(t) = \int_{-\infty}^{\infty} f(s) g(t-s) \, ds \longleftrightarrow F(\omega) G(\omega)$$

**Product Theorem:** If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t) g(t) \longleftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega).$$