The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016–17**

Mathematics Core 1 **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

This exam paper has three sections. Section A and Section B are multiple choice questions. The correct answer, or answers, must be indicated on the exam paper itself.

Answers to Section C must be written on the answer booklet provided.

Total marks: ***55***

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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Section A:

Each question or incomplete statement in this section is followed by five possible options of which exactly one is correct. Mark clearly the correct answer on the question paper.

(17 marks)

A1 If $A = \{1, 2\}$ and $B = \{2, 3\}$, which of the following ordered pairs is not an element of $A \times B$?

A. $(1, 2)$ **B.** $(1, 3)$ **C.** $(2, 2)$ **D.** $(3, 2)$ **E.** $(2, 3)$

A2 How many functions from $\{1, 2, 3, 4\}$ to $\{5, 6, 7\}$ are there?

A. 12 **B.** 64 **C.** 81 **D.** $4! \times 3!$ **E.** $(4!)^3$

A3 How many different rearrangements of the word `STATISTICS` are there?

A. $10!$ **B.** $\dfrac{10!}{3!}$ **C.** $\dfrac{10!}{3!3!2!}$ **D.** $\dfrac{10^{10}}{3^3 3^3 2^2}$ **E.** $5! \cdot (3!)^2 \cdot 2!$

A4 If $r \neq 1$ then $1 + r + r^2 + r^3 + r^4$ is equal to

A. $(1 + r)^4$ **B.** $\dfrac{1 - r^5}{1 + r}$ **C.** $\dfrac{1 + r^5}{1 + r}$ **D.** $\dfrac{1 - r^5}{1 - r}$ **E.** $r^5 - 1$

A5 If $\pi/2 < \theta < \pi$ and $\sin\theta = \dfrac{5}{13}$ then $\sin(2\theta) =$

A. $\dfrac{12}{13}$ **B.** $\dfrac{120}{169}$ **C.** $-\dfrac{120}{169}$ **D.** $\dfrac{60}{169}$ **E.** $-\dfrac{60}{169}$

A6 $\lim_{x\to 1} \dfrac{x^3-1}{x-1} =$

A. 3 **B.** 2 **C.** 1 **D.** 0 **E.** ∞

A7 $\lim_{x\to 0} \dfrac{\sin(5x)}{x} =$

A. 7 **B.** 6 **C.** 5 **D.** 0 **E.** ∞

A8 If $y = \sin(x^3)$ then $\dfrac{dy}{dx} =$

A. $\sin(3x^2)$ **B.** $3\sin(x^2)$ **C.** $\cos(3x^2)$ **D.** $3\cos(x^2)$ **E.** $3x^2\cos(x^3)$

A9 $\dfrac{1+3i}{1-2i} - i =$

A. $2-i$ **B.** -1 **C.** 0 **D.** 1 **E.** $\dfrac{1-6i}{5}$

A10 $e^{i3\pi/4} =$

A. $\dfrac{1+i}{\sqrt{2}}$ **B.** $\dfrac{1-i}{\sqrt{2}}$ **C.** $\dfrac{1}{\sqrt{2}}$ **D.** $\dfrac{-1+i}{\sqrt{2}}$ **E.** $\dfrac{i}{\sqrt{2}}$

A11 If $z \in \mathbb{C}$ has modulus $|z| = 5$, then $|\overline{z}| + \left|\dfrac{z}{\overline{z}}\right| =$

A. 25 **B.** 10 **C.** 6 **D.** 110 **E.** 8

A12 If $\arg(z) = \dfrac{\pi}{3}$ and $\arg(w) = \dfrac{\pi}{4}$, then $\arg\left(\dfrac{z}{w}\right) =$

A. $\dfrac{7\pi}{12}$ **B.** $\dfrac{\pi}{5}$ **C.** $\dfrac{4}{3}$ **D.** $\dfrac{4}{3}$ **E.** $\dfrac{\pi}{12}$

A13 If $\lim_{n\to\infty} a_n = 5$ and $\lim_{n\to\infty} b_n = 2$ then $\lim_{n\to\infty} (a_n b_n + a_{n+1} + b_{n+2}) =$

A. 15 **B.** 16 **C.** 17 **D.** 18 **E.** 19

A14 The sequence generated by the recurrence $a_{n+1} = \sqrt{3 + 2a_n}$ for positive integers n and $a_1 = 1$ is known to be convergent. What is $\lim_{n\to\infty} a_n$?

A. -1 **B.** 0 **C.** $\sqrt{5}$ **D.** 3 **E.** 5

A15 The series $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$ is the Maclaurin series expansion of which function?

A. $\cos x$ **B.** $\sin(x^2)$ **C.** $\sin x$ **D.** $\tan x$ **E.** $e^x \sin x$

A16 The infinite series $1 + 2x + 2^2x^2 + 2^3x^3 + \ldots$ has radius of convergence

A. 1 **B.** 2 **C.** -2 **D.** $\dfrac{1}{2}$ **E.** ∞

A17 $\dfrac{d}{dx}\displaystyle\int_0^x t^{110} \sin(t)\, dt =$

A. $x^{111}\cos x$ **B.** $\dfrac{x^{111}\cos x}{111}$ **C.** $x^{111}\sin x$ **D.** $\dfrac{x^{111}\sin x}{111}$ **E.** $x^{110}\sin x$

Section B:

Each question in this section is followed by six possible options of which exactly two are correct. Mark clearly the correct answers on the question paper. ***(8 marks)***

B1 $f : U \to V$ and $g : V \to W$ are two functions such that the composite $g \circ f : U \to W$ is a bijection. Which two from the following list of statements are true?

A. f is injective. **B.** f is surjective. **C.** f is bijective.

D. g is injective. **E.** g is surjective. **F.** g is bijective.

B2 If $\displaystyle\int e^x \sin x \, dx = e^x(a \sin x + b \cos x)$, what are the values of a and b?

A. $a = 0$ **B.** $a = -\dfrac{1}{2}$ **C.** $a = \dfrac{1}{2}$

D. $b = 0$ **E.** $b = -\dfrac{1}{2}$ **F.** $b = \dfrac{1}{2}$

B3 There are three complex numbers z such that $z^3 = 8$. One of them is $z = 2$. What are the other two?

A. $2i$ **B.** $1 + i\sqrt{3}$ **C.** $-1 + i\sqrt{3}$

D. $-2i$ **E.** $1 - i\sqrt{3}$ **F.** $-1 - i\sqrt{3}$

B4 Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} 0, & \text{when } 0 \leq x \leq 1, \\ x, & \text{when } x < 0 \text{ or } x > 1. \end{cases}$$

Which two from the following list of statements are true?

A. f is continuous at 0. **B.** f is not continuous at 0.

C. f is continuous at 1. **D.** f is not continuous at 1.

E. f is differentiable at 0. **F.** f is differentiable at 1

Section C

C1 Use induction to prove that

$$\frac{1}{2!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

for all $n \in \mathbb{N}$. ***(5 marks)***

C2 Write down the expression for $f'(x)$, the derivative of a function f at x, as a suitable limit. Using this, show that if $f(x) := x^2$ then $f'(x) = 2x$. ***(3 marks)***

C3 Evaluate $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$. ***(3 marks)***

C4 Write down the Maclaurin series expansion for $\dfrac{1}{1-x}$, and show that

$$\frac{2 - 8x}{1 - 8x + 15x^2} = \sum_{n=0}^{\infty} (3^n + 5^n) x^n.$$

You do not need to worry about the radii of convergence of the series involved.
(4 marks)

C5 In this question y is a function of t. Thus $y' = \dfrac{dy}{dt}$ and $y'' = \dfrac{d^2y}{dt^2}$. You should assume that the variable is always positive i.e. $t > 0$.

(i) Find the general solution to the differential equation $y'' - 5y' + 6y = 0$.
(2 marks)

(ii) Find a particular solution to the differential equation $y'' - 5y' + 6y = e^{3t}$.
(3 marks)

C6 (i) Let $0 \leq \theta \leq \dfrac{\pi}{2}$, and let

$$I = \int_0^\theta \frac{\cos x}{\sin x + \cos x}\, dx, \qquad J = \int_0^\theta \frac{\sin x}{\sin x + \cos x}\, dx.$$

Evaluate $I + J$ and $I - J$, and deduce that $2I = \theta + \ln(\sin\theta + \cos\theta)$.

(4 marks)

(ii) Now let a, b be fixed positive real numbers. Show that

$$(a^2 + b^2) \int_0^{\frac{\pi}{2}} \frac{\cos x}{a\sin x + b\cos x}\, dx = \frac{\pi b}{2} + a\ln(a/b).$$

(6 marks)

End of Question Paper