The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn 2016**

Advanced Calculus and Linear Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper E denotes an identity matrix.

1 (i) Define $(x, y) = F(u, v)$ for $u > 0$ and $v \in \mathbb{R}$ by

$$x = \cosh u \cos v, \qquad y = \sinh u \sin v. \tag{1}$$

(a) Find the derivative matrix and the Jacobian of F.

(b) Find the coordinate curve for $u = u_0 > 0$ as an equation in x and y.
(5 marks)

(ii) (a) Let $\iint_R \varphi(x, y)\; dx\, dy$ be a double integral defined on a suitable region R of the x, y–plane. Let $(x, y) = F(u, v)$ be a smooth map defined for $(u, v) \in S$ where S is a suitable region of the u, v–plane.

State, without proof, the formula which expresses $\iint_R \varphi(x, y)\; dx\, dy$ in terms of a double integral over S, including the conditions which F must satisfy. **(5 marks)**

(b) Let R be the region bounded by the circle of radius $a > 0$, centre the origin. Use a substitution to calculate

$$\iint_R e^{-x^2-y^2}\; dx\, dy.$$

(5 marks)

(iii) A set of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ in $\mathbb{R}^p$, each nonzero, is said to be *orthogonal* if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for all $i \neq j$.

(a) Prove that if $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is an orthogonal set of vectors in $\mathbb{R}^p$ then it is linearly independent. **(5 marks)**

(b) Let $u, v, w, t \in \mathbb{R}$ and assume that $u^2 + v^2 + w^2 + t^2 \neq 0$. Using (a) or otherwise, find the rank and the nullity of the matrix

$$\begin{bmatrix} 2t & -2w & -2v & 2u \\ 2v & 2u & 2t & 2w \\ 2u & -2v & 2w & -2t \\ -2w & -2t & 2u & 2v \end{bmatrix},$$

stating clearly, but not proving, any general theorem from the course that you use. **(5 marks)**

2 (i) Let $(x, y, z) = F(u, v, w)$ be a smooth map defined for all $(u, v, w) \in \mathbb{R}^3$ and let $(u, v, w) = \gamma(t)$ be a smooth map defined for $t \in \mathbb{R}$.

(a) State the Chain Rule for $F \circ \gamma$ as a matrix equation. **(2 marks)**

(b) Now let $(x, y, z) = F(r, \theta, \varphi)$ be the spherical polar coordinates map, that is,

$$x = r\cos\theta\cos\varphi, \quad y = r\sin\theta\cos\varphi, \quad z = r\sin\varphi,$$

where $r \geqslant 0$, $0 \leqslant \theta \leqslant 2\pi$, $-\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}$.

Find the derivative matrix $D(F)(r, \theta, \varphi)$. **(3 marks)**

(c) Now let $\gamma(t) = (r(t), \theta(t), \varphi(t))$ be a smooth map defined for $t \in \mathbb{R}$. Using (a) and (b) find

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

(7 marks)

(ii) (a) Let $\mathbf{u} = L(\mathbf{x})$ be a map $L\colon \mathbb{R}^p \to \mathbb{R}^q$. State the conditions for L to be a *linear* map. **(2 marks)**

(b) Given $\mathbf{a} = (a, b, c)$ define a map $L\colon \mathbb{R}^3 \to \mathbb{R}^3$, in terms of the cross product, by

$$L(\mathbf{x}) = \mathbf{a} \times \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3.$$

Show that L is linear. You may use, without proof, standard properties of the cross product. **(2 marks)**

Find the matrix corresponding to L in terms of a, b, c.

Now assume that $\mathbf{a} \neq \mathbf{0}$. Find a non-zero element of the kernel of L, and give a brief justification of your answer. **(4 marks)**

(iii) Let $a, b, c > 0$. By means of a substitution using a modification of spherical polars, or otherwise, calculate the triple integral

$$\iiint_R x \; dx\, dy\, dz$$

where R is the region in the octant $x > 0$, $y > 0$, $z > 0$ bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

You may use without proof, if you wish, the fact that the Jacobian for spherical polars is $r^2\cos\varphi$. **(5 marks)**

3 (i) The surface S has parametric form

$$x(u,v) = u\cos v, \quad y(u,v) = u\sin v, \quad z(u,v) = 3-u,$$

where $1 \leqslant u \leqslant 3$ and $0 \leqslant v < 2\pi$.

(a) The element of surface area $\mathrm{d}S$ is given by

$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}u\ \mathrm{d}v,$$

where $\mathbf{r}(u,v)$ is the position vector of a point in S.
Find $\mathrm{d}S$ for the surface S. **(5 marks)**

(b) Evaluate the surface integral

$$I = \int_S f(x,y,z)\ \mathrm{d}S,$$

where $f(x,y,z) = \dfrac{z}{x^2+y^2}$. **(7 marks)**

(c) Sketch the surface S. **(3 marks)**

(ii) Let $\mathbf{v} = (z^2 - y, 2x + 3z^2, x^2 + y^2)$ and S' be the hemisphere $x^2 + y^2 + z^2 = 4, \quad z \geqslant 0$.

(a) Find curl $\mathbf{v}$. **(2 marks)**

(b) Show that the boundary of S' is a circle in the xy-plane, and write down a parameterisation for this circle. **(2 marks)**

(c) Use Stokes' theorem to convert the surface integral

$$J = \int_{S'} (\text{curl } \mathbf{v}) \cdot \hat{\mathbf{n}}\ \mathrm{d}S$$

to a line integral around the boundary of S'. Hence evaluate the integral J. **(6 marks)**

4 (i) Consider the function $f(x,y) = x^3 - 6x^2 - y^3 + 12y + 4$.

(a) Find grad f. **(2 marks)**

(b) Find and classify all stationary points of $f(x,y)$. Do any of the stationary points correspond to *global* maxima or minima of f? **(8 marks)**

(c) For each stationary point, find the Taylor series for $f(x,y)$ in the form

$$f(a+h, b+k) = f_0 + Ah^2 + 2Bhk + Ck^2 + \ldots,$$

where (a,b) are the coordinates of the stationary point, and f_0, A, B and C are constants, which you should determine. Hence sketch the level sets of $f(x,y)$ in the local vicinity of each stationary point. **(7 marks)**

(ii) Find the maximum and minimum values of the function $f(x,y) = x + 2y$ subject to the constraint $x^2 - 2x + y^2 - 4y = 4$. **(8 marks)**

End of Question Paper