Data provided: formula sheet

MAS241

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Autumn Semester 2016–17**

Mathematics II (Electrical) — **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Consider the system of differential equations

$$\begin{aligned} y'(t) &= y(t) - 2x(t) \\ x'(t) &= 5y(t) - x(t) \end{aligned}$$

subject to the initial conditions $x(0) = 2$ and $y(0) = -1$.

(a) Show that the Laplace transforms of $x(t)$ and $y(t)$ are given by

$$\begin{aligned} X(s) &= \mathcal{L}\{x(t)\} = \frac{2s-7}{s^2+9}, \\ Y(s) &= \mathcal{L}\{y(t)\} = -\frac{s+5}{s^2+9}. \end{aligned}$$

(8 marks)

(b) Find the $x(t)$ and $y(t)$ that satisfy the system of differential equations. **(8 marks)**

(ii) Let $f(t) = H(t)H(1-t)$.

(a) Sketch the graph of $f(t)$ over the interval $[-3, 3]$. **(2 marks)**

(b) Calculate $\mathcal{L}\{f * f(t)\}$. **(2 marks)**

2 Consider the function defined by $f(t) = t$ on $[0, 1]$.

(i) Find the Fourier sine series of $f(t)$. **(14 marks)**

(ii) Sketch the graph of the Fourier sine series that you calculated in Part (a) over the interval $[-2, 3]$. Your sketch should indicate the values of the series at points of discontinuity. **(4 marks)**

(iii) Find an exact value for the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots.$$

(2 marks)

3 (i) Find and classify *all* the critical points of the function

$$f(x,y) \;=\; x^3 - 3xy + y^3.$$

(12 marks)

(ii) For $z = x + jy$, let

$$u(x,y) \;=\; \mathrm{Re}(z^2) \qquad \text{and} \qquad v(x,y) \;=\; \mathrm{Im}(z^2)$$

be the real and imaginary parts of the complex function $f(z) = z^2$. Show that

$$u_x(x,y) \;=\; v_y(x,y) \qquad \text{and} \qquad u_y(x,y) \;=\; -\,v_x(x,y).$$

(4 marks)

(iii) Let $x(u,v) = u - v$ and $y(u,v) = u + v$. Show that if $f(x,y)$ has continuous second order derivatives then

$$f_{uv} \;=\; f_{yy} - f_{xx}.$$

(4 marks)

4 (i) Let $T \subset \mathbb{R}^2$ be the region bounded by the lines $y = x$, $y = -x$, and $y = 1$, and let $f(x,y) = x^2 y$.

(a) Sketch the region T. **(2 marks)**

(b) Find

$$\iint_T f(x,y)\; dA.$$

(8 marks)

(ii) Let D be the region in the first quadrant of $\mathbb{R}^2$ (i.e. $x, y > 0$) bounded by the four curves described by $y = x^2$, $y = 3x^2$, $y = \dfrac{1}{x}$, and $y = \dfrac{3}{x}$.

(a) Sketch the region D. **(2 marks)**

(b) Find the area of D. **(8 marks)**

5 (i) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F} = \left(xy^2z^3, e^{xy+yz+xz}, \sin(xyz)\right).$$

(a) Calculate **div F**. **(5 marks)**

(b) Calculate **curl F**. **(5 marks)**

(ii) Calculate the directional derivative of the function $f(x, y) = x^2y^2 + 2xy + 1$ at $(1, 2)$ in the direction $\mathbf{v} = (1, -1)$. **(5 marks)**

(iii) Sketch the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{F} = \left(-\frac{x}{2}, -\frac{y}{2}\right).$$

(5 marks)

End of Question Paper

MAS241 FORMULA SHEET

Laplace transform:

The Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t) dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k + s)$	frequency shift
$\mathcal{L}\{f(t - a)H(t - a)\} = e^{-as}F(s) \quad$ (for $a > 0$)	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \quad$ (for $a > 0$)	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
$t^n \quad$ (for $n \geq 0$)	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$H(t - T) \quad$ (for $T \geq 0$)	$\frac{e^{-sT}}{s}$	$Re(s) > 0$
$\delta(t - T) \quad$ (for $T \geq 0$)	e^{-sT}	$s \in \mathbb{C}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{e^{j\theta t}f(t)\} = F(\omega - \theta)$	frequency shift
$\mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$e^{-a\lvert t\rvert}$ (for $a > 0$)	$\frac{2a}{a^2+\omega^2}$
$\text{rect}_T(t)$	$\text{sinc}(\frac{T\omega}{2})$
1	$2\pi\delta(\omega)$

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)dt.$$

Coordinate systems:

Cylindrical polar coordinates

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$
$$(r, \theta, z) = \Big(\sqrt{x^2+y^2}, \arctan(\tfrac{y}{x}), z\Big)$$
$$dV = r dr d\theta dz.$$

Spherical polar coordinates

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$
$$(\rho, \theta, \phi) = \Big(\sqrt{x^2+y^2+z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho})\Big)$$
$$dV = \rho^2\sin(\phi)d\rho d\phi d\theta.$$