Data provided: Formula sheet

The University Of Sheffield.

MAS248

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016–17**

MATHEMATICS III (CHEMICAL) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets. The paper is marked out of a total of 60 marks.

1 (i) A force field $\mathbf{F}(x, y, z)$ is given by

$$\mathbf{F} = (\sin y + z, x\cos y - z, x - y).$$

(a) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$. ***(2 marks)***

(b) Find a scalar potential V such that $\mathbf{F} = \nabla V$. ***(5 marks)***

(c) Compute the divergence of $\mathbf{F}$. ***(1 mark)***

(d) Calculate $\nabla^2 V$. ***(2 marks)***

(ii) If $\mathbf{H}$ is a constant vector, prove that

$$\nabla(\mathbf{H} \cdot \mathbf{r}) = \mathbf{H},$$

where $\mathbf{r}$ is the position vector field (x, y, z). ***(2 marks)***

(iii) Find the directional derivative of the scalar field $q(x, y, z)$ which is given by

$$q = x^3 + yz$$

at the point $(2, 3, 3)$ in the direction of the vector $\mathbf{a} = (1, 2, 2)$. ***(3 marks)***

2 (i) Find and classify the stationary points of the function

$$f(x, y) = x^3 + 15x^2 - 20y^2 + 10.$$

(7 marks)

(ii) A biased die has probabilities p, $2p$, $3p$, $4p$, $5p$ and $6p$ of throwing the numbers 1, 2, 3, 4, 5 and 6, respectively.

(a) Find p. ***(2 marks)***

(b) What is the probability of throwing an even number? ***(2 marks)***

(iii) A continuous random variable X has probability density function

$$p\,(x) = \begin{cases} cxe^{-2x} & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \leq 0. \end{cases}$$

Find the value of the constant c. ***(4 marks)***

3 (i) Write down the iteration formula for the Newton-Raphson method. Starting with an initial guess of $x_0 = -4.0$, use the Newton-Raphson method to find the root of the function

$$f(x) = x^3 + 4x^2 + 7$$

that is in the vicinity of $x = -4.0$ correct to 5 decimal places. ***(4 marks)***

(ii) A periodic function, $f(t)$, with period 2π is defined by

$$f(t) = t^2 \quad \text{for} \quad -\pi \leq t < \pi.$$

(a) Sketch a graph of the function $f(t)$ for values of t from $t = -4\pi$ to $t = 4\pi$. ***(2 marks)***

(b) Show that the Fourier series for $f(t)$ in the interval $-\pi \leq t < \pi$ is given by

$$f(t) = \frac{\pi^2}{3} - 4\left(\cos t - \frac{1}{2^2}\cos 2t + \frac{1}{3^2}\cos 3t - \frac{1}{4^2}\cos 4t + \cdots\right).$$

(7 marks)

(c) By giving an appropriate value to t show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

(2 marks)

4 Show that the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} + 6\frac{\partial^2 y}{\partial x \partial t} - 7\frac{\partial^2 y}{\partial x^2} = 0$$

has solutions of the form $y(x,t) = f(x + \lambda t)$ for arbitrary twice differentiable functions f, provided that $\lambda = -7$ or $\lambda = 1$.

Write down the general solution of the partial differential equation. ***(5 marks)***

Find the particular solution for y that satisfies the conditions

$$y(x,0) = 0$$

and

$$\frac{\partial y}{\partial t}(x,0) = 7x.$$

(10 marks)

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 0, 1, 2, ...,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \leq x \leq L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Gradient of a Scalar Field

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla \phi = \text{grad } \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x, y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x, y)$ has a stationary point at (x_0, y_0) if

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0).$$

2 At (x_0, y_0), the function $f(x, y)$ has:

(i) a minimum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \quad (x_0, y_0),$$

(ii) a maximum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \quad (x_0, y_0),$$

(iii) a saddle point if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$