SCHOOL OF MATHEMATICS AND STATISTICS  

Mathematics II (Materials)  

Autumn Semester 2016–17

2 hours

Marks will be awarded for answers to all questions in Section A, and for your best THREE answers to questions in Section B. Section A carries 40 marks, and the marks awarded to each question or section of question are shown in italics. The maximum possible mark for the paper is 100.

Section A

A1  Find the particular solution of the equation

\[ x^2 \frac{dy}{dx} + y = \pi x^2 e^{\pi x} \cos \pi x \quad (x > 0) \]

which satisfies \( y = 0 \) when \( x = 1 \).  (9 marks)

A2  Find the general solution of the equation

\[ \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = e^x. \]  (7 marks)

A3  A ball of mass 0.8 kg is thrown at a speed of 20 ms\(^{-1}\). The kinetic energy of the ball is

\[ E = \frac{1}{2}mv^2, \]

where \( m \) and \( v \) are its mass (in kg) and velocity (in ms\(^{-1}\)), respectively.

A ball of mass 0.84 kg is now thrown at a speed of 19.2 ms\(^{-1}\). Use the chain rule for partial derivatives to estimate the % change in the kinetic energy from the first throw.  (7 marks)
A4 A scalar field \( \phi \) is given by
\[
\phi = 3xy^2 + 2yz^2 + zx^2,
\]
and a vector field \( u \) is defined by
\[
u = \nabla \phi.
\]

(a) Find \( u \), \( \nabla \cdot u \) and \( \nabla \times u \). \hfill (6 marks)

(b) Find the directional derivative of \( \phi \) in the direction of the vector \( (1, 1, -1) \) at the point \( (1, -2, 3) \). \hfill (5 marks)

A5 A die is rolled five times, giving the numbers
6, 1, 3, 4, 1.

Showing your working, find the mean, standard deviation and skewness
\[
\left( = \frac{1}{n} \sum(x - \bar{x})^3 \right)
\]
of these results, giving your answers to 3 significant figures. \hfill (6 marks)

Section B

B1 (a) For \( x > 0 \), find the general solution of the equation
\[
\frac{dy}{dx} = \frac{x^2 + 2y^2}{xy}.
\]
\hfill (8 marks)

(b) Find the general solution of the equation
\[
2 \frac{d^2y}{dx^2} \frac{dy}{dx} - 3y = x + e^{-x}.
\]
\hfill (12 marks)
B2  (a) The following table shows the wage bills (in millions of pounds) and the final points totals of 10 Premier League football clubs for 2007-08:

<table>
<thead>
<tr>
<th>Club</th>
<th>Wages (x)</th>
<th>Points (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arsenal</td>
<td>101.3</td>
<td>83</td>
</tr>
<tr>
<td>Aston Villa</td>
<td>50.4</td>
<td>60</td>
</tr>
<tr>
<td>Blackburn Rovers</td>
<td>39.7</td>
<td>58</td>
</tr>
<tr>
<td>Bolton Wanderers</td>
<td>39</td>
<td>37</td>
</tr>
<tr>
<td>Chelsea</td>
<td>149</td>
<td>85</td>
</tr>
<tr>
<td>Everton</td>
<td>44.5</td>
<td>65</td>
</tr>
<tr>
<td>Fulham</td>
<td>39.3</td>
<td>36</td>
</tr>
<tr>
<td>Liverpool</td>
<td>80</td>
<td>76</td>
</tr>
<tr>
<td>Manchester City</td>
<td>54.2</td>
<td>55</td>
</tr>
<tr>
<td>Manchester United</td>
<td>121.1</td>
<td>87</td>
</tr>
</tbody>
</table>

Showing your working, calculate the means and variances of $x$ and $y$, and also the correlation between $x$ and $y$, giving the correlation to 3 significant figures.  

(12 marks)

Comment briefly on the implications of the correlation between $x$ and $y$.  

(1 mark)

(b) A vector field $u$ is given by

$$u = (e^{xy} + z \cos y, xyz + \sin(xy), x \cosh z - yz).$$

Verify that

$$\nabla \cdot (\nabla \times u) = 0.$$  

(7 marks)

B3  The function $f(x) = x^2$ is defined on the interval $0 \leq x \leq 1$.

(a) Show that $f(x)$ can be represented by the Fourier series

$$\frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{n \pi x}{n} - \frac{8}{\pi^3} \sum_{m=1}^{\infty} \frac{\sin(2m-1) \pi x}{(2m-1)^3}.$$  

(17 marks)

(b) Sketch the function given by the above Fourier series on the interval $-3 \leq x \leq 3$.  

(3 marks)
The temperature \( u(x, t) \) (in units of °C) satisfies the heat conduction equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\]

on \( 0 \leq x \leq 1 \), for \( t > 0 \), subject to the boundary conditions

\[
u = 50 \quad \text{at } x = 0
\]

and \( u = 0 \) at \( x = 1 \).

Show that the general solution is

\[
 u(x, t) = 50(1 - x) + \sum_{n=1}^{\infty} B_n \exp \left( -n^2 \pi^2 t \right) \sin (n\pi x),
\]

where the \( B_n \) are undetermined constants. \( \text{(17 marks)} \)

What kind of condition would enable you to find the values of the \( B_n \), and what type of method would you use to find them? \( \text{(3 marks)} \)

End of Question Paper
FORMULA SHEET

Trigonometry

\[ 1 + \tan^2 \theta = \sec^2 \theta \]
\[ 1 + \cot^2 \theta = \cosec^2 \theta \]
\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \]
\[ \cos(A - B) = \cos A \cos B + \sin A \sin B \]
\[ \sin(A + B) = \sin A \cos B + \cos A \sin B \]
\[ \sin(A - B) = \sin A \cos B - \cos A \sin B \]
\[ \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \]
\[ \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \]
\[ \sin 2\theta = 2 \sin \theta \cos \theta \]
\[ \cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \]
\[ a \cos \theta + b \sin \theta = R \cos(\theta - \alpha), \text{ where } R = \sqrt{(a^2 + b^2)}, \cos \alpha = a/R \text{ and } \sin \alpha = b/R \]

Hyperbolic Functions

\[ \sinh x = \frac{1}{2}(e^x - e^{-x}) \]
\[ \cosh x = \frac{1}{2}(e^x + e^{-x}) \]
\[ \cosh^2 x - \sinh^2 x = 1 \]
\[ \text{sech}^2 x + \tanh^2 x = 1 \]
\[ 2 \sinh x \cosh x = \sinh 2x \]
\[ \cosh 2x = 2 \cosh^2 x - 1 = 2 \sinh^2 x + 1 \]
\[ \sinh^{-1} x = \ln \left[ x + \sqrt{(1 + x^2)} \right], \text{ all } x \]
\[ \cosh^{-1} x = \ln \left[ x + \sqrt{(x^2 - 1)} \right], \text{ } x \geq 1 \]
\[ \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right), \text{ } |x| < 1 \]
\[
\coth^{-1} x = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right), \quad |x| > 1
\]
### Differentiation and Integration

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$\ln x$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>$\sec^2 x$</td>
</tr>
<tr>
<td>$\cot x$</td>
<td>$-\csc^2 x$</td>
</tr>
<tr>
<td>$\sec x$</td>
<td>$\sec x \tan x$</td>
</tr>
<tr>
<td>$\csc x$</td>
<td>$-\csc x \cot x$</td>
</tr>
<tr>
<td>$\sinh x$</td>
<td>$\cosh x$</td>
</tr>
<tr>
<td>$\cosh x$</td>
<td>$\sinh x$</td>
</tr>
<tr>
<td>$\tanh x$</td>
<td>$\operatorname{sech}^2 x$</td>
</tr>
<tr>
<td>$\coth x$</td>
<td>$-\operatorname{csch}^2 x$</td>
</tr>
<tr>
<td>$\operatorname{sech} x$</td>
<td>$-\operatorname{sech} x \tanh x$</td>
</tr>
<tr>
<td>$\operatorname{cosech} x$</td>
<td>$-\operatorname{cosech} x \coth x$</td>
</tr>
<tr>
<td>$\sin^{-1} x$</td>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
</tr>
<tr>
<td>$\cos^{-1} x$</td>
<td>$-\frac{1}{\sqrt{1-x^2}}$</td>
</tr>
<tr>
<td>$\tan^{-1} x$</td>
<td>$\frac{1}{1+x^2}$</td>
</tr>
<tr>
<td>$\cot^{-1} x$</td>
<td>$-\frac{1}{1+x^2}$</td>
</tr>
<tr>
<td>$\sinh^{-1} x$</td>
<td>$\frac{1}{\sqrt{x^2+1}}$</td>
</tr>
<tr>
<td>$\cosh^{-1} x$</td>
<td>$\frac{1}{\sqrt{x^2-1}}$</td>
</tr>
<tr>
<td>$\tanh^{-1} x$</td>
<td>$\frac{1}{1-x^2}$, $</td>
</tr>
<tr>
<td>$\coth^{-1} x$</td>
<td>$-\frac{1}{x^2-1}$, $</td>
</tr>
<tr>
<td>Function</td>
<td>Integral</td>
</tr>
<tr>
<td>--------------------------------------</td>
<td>---------------------------------------</td>
</tr>
<tr>
<td>( \frac{1}{a^2 + x^2} )</td>
<td>( \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) )</td>
</tr>
<tr>
<td>( \frac{1}{a^2 - x^2} )</td>
<td>( \frac{1}{a} \tanh^{-1} \left( \frac{x}{a} \right) )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{a^2 - x^2}} )</td>
<td>( \sin^{-1} \left( \frac{x}{a} \right) )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{a^2 + x^2}} )</td>
<td>( \sinh^{-1} \left( \frac{x}{a} \right) )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{x^2 - a^2}} )</td>
<td>( \cosh^{-1} \left( \frac{x}{a} \right) )</td>
</tr>
</tbody>
</table>

Differentiation and Integration Formulae

\[
\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}
\]

\[
\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]

\[
\int_a^b uv \, dx = \left[ u \times \text{(integral of } v) \right]_a^b - \int_a^b \frac{du}{dx} \times \text{(integral of } v) \, dx
\]

Partial Differentiation

Chain Rule

1. Suppose that \( z = f(x, y) \) and that \( x \) and \( y \) are functions of \( t \), i.e., \( x = x(t) \), \( y = y(t) \). Then

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

2. Suppose that \( z = f(x, y) \) and that \( x \) and \( y \) are functions of the variables \( r \) and \( s \), i.e., \( x = x(r, s) \), \( y = y(r, s) \). Then

\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}, \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]
First-Order Differential Equations

1. Direct Integration

\[
\frac{dy}{dx} = f(x)
\]

\[y = \int f(x) \, dx + C\]

2. Separation of Variables

\[
\frac{dy}{dx} = f(x)g(y)
\]

\[
\int \frac{dy}{g(y)} = \int f(x) \, dx
\]

3. Homogeneous Equations

\[
\frac{dy}{dx} = f \left(\frac{y}{x}\right)
\]

make the substitution \( y = zx \) to give

\[z + x \frac{dz}{dx} = f(z)\]

4. Linear Equations

\[
\frac{dy}{dx} + P(x)y = Q(x)
\]

multiply both sides by the integrating factor \( e^{\int P(x) \, dx} \) to give

\[
\frac{d}{dx} \left( ye^{\int P(x) \, dx} \right) = Q(x) e^{\int P(x) \, dx}\]
The Second-Order Differential Equation

\[ a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \]

where \( a, b, \) and \( c \) are constants.

General solution is

\[ y = \text{Complementary Function} + \text{Particular Integral} \]

The solution, \( y_c \), is given by

(i) \( y_c = Ae^{m_1x} + Be^{m_2x} \), if \( m_1 \) and \( m_2 \) real and different,

(ii) \( y_c = e^{mx}(A + Bx) \), if \( m_1 \) and \( m_2 \) real and equal \( (m_1 = m_2 = m) \),

(iii) \( y_c = e^{px}(A \cos qx + B \sin qx) \), if \( m_1 \) and \( m_2 \) are complex \( (m_1 = p + iq, m_2 = p - iq) \),
where \( m_1 \) and \( m_2 \) are the roots of the auxiliary equation

\[ am^2 + bm + c = 0 \]

Particular Integral, \( y_p \)

\[ f(x) = Ax^2 + Bx + C \quad y_p = ax^2 + bx + c \]

\[ f(x) = Ae^{kx} \quad y_p = ae^{kx} \]
when \( k \) is not one of the roots of the auxiliary equation

\[ f(x) = Ae^{kx} \quad y_p = axe^{kx} \]
when \( k \) is one of the roots of the auxiliary equation

\[ f(x) = A \cos mx + B \sin mx \quad y_p = a \cos mx + b \sin mx \]
when \( \sin mx \) or \( \cos mx \) is not part of the complementary function

\[ f(x) = A \cos mx + B \sin mx \quad y_p = x(a \cos mx + b \sin mx) \]
when \( \sin mx \) or \( \cos mx \) is part of the complementary function
Fourier Series

Suppose that \( f(x) \) is defined on the interval \(-l \leq x \leq l\). The Fourier series for \( f(x) \) is given by

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right),
\]

where

\[
a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad n = 0, 1, 2, \ldots,
\]

\[
b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx, \quad n = 1, 2, \ldots.
\]

On the interval \( 0 \leq x \leq l \) the Fourier cosine series for \( f(x) \) is

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \quad a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l} \, dx
\]

and the Fourier sine series is

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx.
\]

Vector Calculus

The gradient of the scalar field \( \phi(x, y, z) \) is given by

\[
\nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).
\]

The divergence of a vector field \( \mathbf{u}(x, y, z) = (u, v, w) \) is given by

\[
\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\]

The curl of a vector field \( \mathbf{u}(x, y, z) = (u, v, w) \) is given by

\[
\nabla \times \mathbf{u} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z}
\end{vmatrix}
\]

The Laplacian \( \nabla^2 \) is given by

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]
Statistics

For data values \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\)

\[
\text{Means } \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \text{ etc.}
\]

\[
\text{Variances } s_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i^2) - \bar{x}^2 \text{ etc.}
\]

\(s_x\) is standard deviation

\[
\text{Covariance } \text{cov}(x, y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n} \sum_{i=1}^{n} (x_i y_i) - \bar{x}\bar{y}
\]

\[
\text{Correlation coefficient } r = \frac{\text{cov}(x, y)}{s_x s_y}
\]

**Linear regression by least squares**

The least squares fit to the linear relationship

\[
y = a + b(x - \bar{x})
\]

is given by

\[
a = \bar{y}, \quad b = \frac{\text{cov}(x, y)}{s_x^2}
\]

The corresponding mean square residual is \(s_y^2(1 - r^2)\).