Data provided: Formula Sheet

MAS250

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2016–17

Mathematics II (Materials) — 2 hours

Marks will be awarded for answers to all questions in Section A, and for your best THREE answers to questions in Section B. Section A carries 40 marks, and the marks awarded to each question or section of question are shown in italics.
The maximum possible mark for the paper is 100.

Section A

A1 Find the particular solution of the equation

$$x^2 \frac{dy}{dx} + y = \pi x^2 e^{1/x} \cos \pi x \qquad (x > 0)$$

which satisfies $y = 0$ when $x = 1$. ***(9 marks)***

A2 Find the general solution of the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x.$$ ***(7 marks)***

A3 A ball of mass 0.8 kg is thrown at a speed of 20 ms^{-1}. The kinetic energy of the ball is

$$E = \frac{1}{2}mv^2,$$

where m and v are its mass (in kg) and velocity (in ms^{-1}), respectively.

A ball of mass 0.84 kg is now thrown at a speed of 19.2 ms^{-1}. Use the chain rule for partial derivatives to estimate the % change in the kinetic energy from the first throw. ***(7 marks)***

A4 A scalar field ϕ is given by

$$\phi = 3xy^2 + 2yz^2 + zx^2,$$

and a vector field $\mathbf{u}$ is defined by

$$\mathbf{u} = \nabla\phi.$$

(a) Find $\mathbf{u}$, $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$. ***(6 marks)***

(b) Find the directional derivative of ϕ in the direction of the vector $(1, 1, -1)$ at the point $(1, -2, 3)$. ***(5 marks)***

A5 A die is rolled five times, giving the numbers

$$6, 1, 3, 4, 1.$$

Showing your working, find the mean, standard deviation and skewness $\left(= \frac{\frac{1}{n}\sum(x-\overline{x})^3}{s^3} \right)$ of these results, giving your answers to 3 significant figures.

(6 marks)

Section B

B1 (a) For $x > 0$, find the general solution of the equation

$$\frac{dy}{dx} = \frac{x^2 + 2y^2}{xy}.$$ ***(8 marks)***

(b) Find the general solution of the equation

$$2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = x + e^{-x}.$$ ***(12 marks)***

B2 (a) The following table shows the wage bills (in millions of pounds) and the final points totals of 10 Premier League football clubs for 2007-08:

Club	Wages (x)	Points (y)
Arsenal	101.3	83
Aston Villa	50.4	60
Blackburn Rovers	39.7	58
Bolton Wanderers	39	37
Chelsea	149	85
Everton	44.5	65
Fulham	39.3	36
Liverpool	80	76
Manchester City	54.2	55
Manchester United	121.1	87

Showing your working, calculate the means and variances of x and y, and also the correlation between x and y, giving the correlation to 3 significant figures. ***(12 marks)***

Comment briefly on the implications of the correlation between x and y. ***(1 mark)***

(b) A vector field **u** is given by

$$\mathbf{u} = (e^{xy} + z\cos y, xyz + \sin(xy), x\cosh z - yz).$$

Verify that

$$\nabla\cdot(\nabla\times\mathbf{u}) = 0.$$ ***(7 marks)***

B3 The function $f(x) = x^2$ is defined on the interval $0 \leqslant x \leqslant 1$.

(a) Show that $f(x)$ can be represented by the Fourier series

$$\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\pi x}{n} - \frac{8}{\pi^3}\sum_{m=1}^{\infty}\frac{\sin(2m-1)\pi x}{(2m-1)^3}.$$ ***(17 marks)***

(b) Sketch the function given by the above Fourier series on the interval $-3 \leqslant x \leqslant 3$. ***(3 marks)***

B4 The temperature $u(x,t)$ (in units of °C) satisfies the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on $0 \leqslant x \leqslant 1$, for $t > 0$, subject to the boundary conditions

$$u = 50 \quad \text{at } x = 0$$
$$\text{and} \quad u = 0 \quad \text{at } x = 1.$$

Show that the general solution is

$$u(x,t) = 50(1-x) + \sum_{n=1}^{\infty} B_n \exp\left(-n^2\pi^2 t\right) \sin\left(n\pi x\right),$$

where the B_n are undetermined constants. ***(17 marks)***

What kind of condition would enable you to find the values of the B_n, and what type of method would you use to find them? ***(3 marks)***

End of Question Paper

FORMULA SHEET

Trigonometry

$1+\tan^2\theta=\sec^2\theta$

$1+\cot^2\theta=\mathrm{cosec}^2\theta$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

$\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\sin(A-B)=\sin A\cos B-\cos A\sin B$

$\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$\sin 2\theta=2\sin\theta\cos\theta$

$\cos 2\theta=2\cos^2\theta-1=1-2\sin^2\theta$

$a\cos\theta+b\sin\theta=R\cos(\theta-\alpha)$, where $R=\sqrt{(a^2+b^2)}$, $\cos\alpha=a/R$ and $\sin\alpha=b/R$

Hyperbolic Functions

$\sinh x=\dfrac{1}{2}(e^x-e^{-x})$

$\cosh x=\dfrac{1}{2}(e^x+e^{-x})$

$\cosh^2 x-\sinh^2 x=1$

$\mathrm{sech}^2 x+\tanh^2 x=1$

$2\sinh x\cosh x=\sinh 2x$

$\cosh 2x=2\cosh^2 x-1=2\sinh^2 x+1$

$\sinh^{-1} x=\ln\left[x+\sqrt{(1+x^2)}\right],\quad \text{all } x$

$\cosh^{-1} x=\ln\left[x+\sqrt{(x^2-1)}\right],\quad x\geqslant 1$

$\tanh^{-1} x=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right),\quad |\,x\,|<1$

$$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad |\, x \,| > 1$$

Differentiation and Integration

Function	Derivative
x^n	nx^{n-1}
$\ln x$	$\dfrac{1}{x}$
e^x	e^x
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec} x$	$-\text{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{cosech}^2 x$
$\text{sech} x$	$-\text{sech} x \tanh x$
$\text{cosech} x$	$-\text{cosech} x \coth x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{(1-x^2)}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{(1-x^2)}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{(x^2+1)}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{(x^2-1)}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}, \quad \mid x \mid < 1$
$\coth^{-1} x$	$-\dfrac{1}{x^2-1}, \quad \mid x \mid > 1$

Function	Integral
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$

Differentiation and Integration Formulae

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\int_a^b uv\,dx = [u \times (\text{integral of } v)]_a^b - \int_a^b \frac{du}{dx} \times (\text{integral of } v)\,dx$$

Partial Differentiation

Chain Rule

1. Suppose that $z = f(x, y)$ and that x and y are functions of t, i.e., $x = x(t)$, $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2. Suppose that $z = f(x, y)$ and that x and y are functions of the variables r and s, i.e., $x = x(r, s)$, $y = y(r, s)$. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

First-Order Differential Equations

1. **Direct Integration**

$$\frac{dy}{dx} = f(x)$$

$$y = \int f(x)dx + C$$

2. **Separation of Variables**

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

3. **Homogeneous Equations**

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

make the substitution $y = zx$ to give

$$z + x\frac{dz}{dx} = f(z)$$

4. **Linear Equations**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

multiply both sides by the integrating factor $e^{\int P(x)dx}$ to give

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

The Second-Order Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b, and c are constants.

General solution is

$$y = \text{Complementary Function} + \text{Particular Integral}$$

The solution, y_c, is given by

(i) $y_c = Ae^{m_1 x} + Be^{m_2 x}$, if m_1 and m_2 real and different,

(ii) $y_c = e^{mx}(A + Bx)$, if m_1 and m_2 real and equal ($m_1 = m_2 = m$),

(iii) $y_c = e^{px}(A\cos qx + B\sin qx)$, if m_1 and m_2 are complex ($m_1 = p + iq$, $m_2 = p - iq$), where m_1 and m_2 are the roots of the *auxiliary equation*

$$am^2 + bm + c = 0$$

Particular Integral, y_p

$f(x) = Ax^2 + Bx + C$ $\quad$ $y_p = ax^2 + bx + c$

$f(x) = Ae^{kx}$ $\quad$ $y_p = ae^{kx}$
when k is not one of the roots of the auxiliary equation

$f(x) = Ae^{kx}$ $\quad$ $y_p = axe^{kx}$
when k is one of the roots of the auxiliary equation

$f(x) = A\cos mx + B\sin mx$ $\quad$ $y_p = a\cos mx + b\sin mx$
when $\sin mx$ or $\cos mx$ is not part of the complementary function

$f(x) = A\cos mx + B\sin mx$ $\quad$ $y_p = x(a\cos mx + b\sin mx)$
when $\sin mx$ or $\cos mx$ is part of the complementary function

Fourier Series

Suppose that $f(x)$ is defined on the interval $-l \leqslant x \leqslant l$. The Fourier series for $f(x)$ is given by

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} \;+\; b_n \sin\frac{n\pi x}{l}\right) \;,$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \;,\quad n = 0, 1, 2, \ldots\,,$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \;,\qquad n = 1, 2, \ldots\,.$$

On the interval $0 \leqslant x \leqslant l$ the Fourier cosine series for $f(x)$ is

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\;,\quad a_n \;=\; \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

and the Fourier sine series is

$$f(x) \;=\; \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\;,\quad b_n \;=\; \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\;.$$

Vector Calculus

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi \;=\; \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)\;.$$

The divergence of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The curl of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The Laplacian ∇^2 is given by

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Statistics

For data values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Means $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ etc.

Variances $s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i^2) - \bar{x}^2$ etc.

s_x is standard deviation

Covariance $\mathrm{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n}\sum_{i=1}^{n}(x_i y_i) - \bar{x}\bar{y}$

Correlation coefficient $r = \frac{\mathrm{cov}(x, y)}{s_x s_y}$

Linear regression by least squares

The least squares fit to the linear relationship

$$y = a + b(x - \bar{x})$$

is given by

$$a = \bar{y}, \qquad b = \frac{\mathrm{cov}(x, y)}{s_x^2}$$

The corresponding mean square residual is $s_y^2(1 - r^2)$.