Data provided: Formula sheet

MAS252

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016–17**

Further Civil Engineering Mathematics and Computing **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 Find the Fourier series representation of the function

$$f(x) = 1 + x\sin 2x,$$

in the interval $-\pi \le x \le \pi$. Discuss separately all cases where the series representation is singular. ***(30 marks)***

2 Use the method of separation of variables to find the solution of the heat conduction equation

$$K\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad u = u(x,t), \quad 0 \le x \le L, \quad K > 0,$$

subject to the boundary conditions

$$u(0,t) = 20, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \quad t > 0,$$

and the initial condition

$$u(x,0) = u_0\frac{x}{L}, \quad 0 \le x \le L$$

(32 marks)

3 (i) If $f(x,y) = y/x$, prove that

$$\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} = 0,$$

and

$$\left(\frac{\partial f}{\partial y}\right)^2 \frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^2 \frac{\partial^2 f}{\partial y^2} = 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y}.$$

(4 marks)

(ii) Use the generalized chain rule to show that if $z = f(x,y)$ and

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv,$$

the identity

$$u\frac{\partial z}{\partial v} - v\frac{\partial z}{\partial u} = 2\left(x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}\right)$$

is valid. ***(7 marks)***

(iii) The volume of a segment of a sphere is given by

$$V = \frac{1}{6}\pi h(h^2 + 3R^2),$$

where R is the radius of the base and h is the height of the segment. If the error in the measurement of the height and radius are α and $-\alpha$, use the small error formula to calculate the error in measurement of the volume.

(6 marks)

4 (i) Values of $y(x)$ at $x = 2$ determined using the fourth-order Runge-Kutta method in conjunction with an ordinary differential equation with two different step-lengths (h) are given in the following table

h	$y(2)$
0.2	3.40978
0.4	3.39278

Use this data to estimate a value for h which will ensure that the error in the calculated value of $y(2)$ using a fourth-order Runge-Kutta method does not exceed 10^{-4}. Give your answer correct to three decimal places.

(6 marks)

(ii) Derive the first four non-zero terms of the Taylor series solution to the differential equation

$$y' = y^3 x + \frac{x^2}{y} - 4, \quad y = y(x)$$

subject to the initial condition $y(1) = 1$. Use this expansion to calculate the value of $y(1.1)$. Give your answer correct to four decimal places.

(15 marks)

End of Question Paper

Formula sheet

- Trigonometric identities

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

- The local truncation error in the case of the 4th order Runge-Kutta method is given by

$$Y(x) - y(x) = Ch^4$$

where $Y(x)$ is the exact value, $y(x)$ is the estimated numerical value, C is a constant and h is the step size used in the numerical scheme.

- **Chain rule**

If $z = f(x, y)$, where x and y are both functions of t, so that $x = x(t)$ and $y = y(t)$ we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$ and both x and y are functions of u and v, so that $x = x(u, v)$ and $y = y(u, v)$ then we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- **Fourier series**

If the function $f(x)$ is defined over the interval $-l \leq x \leq l$, then the Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

If the function $f(x)$ is defined over the interval $0 \leq x \leq l$, then the Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

while the sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\;dx \quad (n = 1, 2, 3, \dots)$$

- the orthogonality of the sine function can be defined as

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } m \neq n \\ L/2 & \text{if } m = n \end{cases}$$