Data provided: Formula sheet

The University Of Sheffield.

MAS253

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016-2017**

Mathematics for Engineering Modelling

2 hours

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Find an expression for the sum of the infinite series

$$4 - 2x^{1/3} + x^{2/3} - \frac{x}{2} + \frac{x^{4/3}}{4} - \frac{x^{5/3}}{8} + ...$$

and determine its radius of convergence, R. Hence find the sum of the infinite series

$$-\frac{2}{3}x^{-2/3} + \frac{2}{3}x^{-1/3} - \frac{1}{2} + \frac{x^{1/3}}{3} - \frac{5}{24}x^{2/3} + ...$$

(7 marks)

(ii) (a) Find the Maclaurin series for

$$f(x) = \frac{2}{8-x}$$

up to terms involving x^3. Use your expression to find an approximate value for $f(7.2)$. Give your answer to 4 significant figures.

(3 marks)

(b) Find the Taylor series for the above $f(x)$ about the point $x = 7$, up to terms involving x^3. Use your result to find another approximate value for $f(7.2)$. Give your answer to 4 significant figures.

(3 marks)

(c) Without comparing with the true value of $f(7.2)$, which approximate value do you think is closer to the true value? Briefly give your reasons. ***(2 marks)***

(iii) Evaluate the following limits:

(a) $$\lim_{x \to -2} \frac{x^2 - x - 6}{3\cos(x+2)\ln(x+3)},$$

(b) $$\lim_{x \to +\infty} x^{\frac{1}{2x}}.$$

(10 marks)

2 Consider the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \cos x & 0 \leq x \leq \pi \end{cases}$$

and its Fourier series denoted by $F(x)$.

(i) Sketch $F(x)$ in the range $-3\pi < x < 3\pi$ and determine the value of $F(0)$. ***(3 marks)***

(ii) Show that

$$F(x) = \frac{1}{2}\cos x + \sum_{m=1}^{\infty} \frac{4m}{\pi(4m^2-1)} \sin(2mx).$$

(17 marks)

(iii) Using a suitable choice for x, deduce that

$$\frac{\pi\sqrt{2}}{16} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k-1)}{4(2k-1)^2-1}.$$

(5 marks)

3 (i) Calculate *by integration* the Laplace transform of $f(t) = e^{-4t}t$. Hence find the Laplace transform of $2e^{-3t}t$. ***(5 marks)***

(ii) With the aid of the Table of Laplace transforms, find the inverse Laplace transforms of

$$\text{(a)}\ \frac{s+3}{4s^2+4s+5}, \quad \text{(b)}\ \frac{e^{-2s}}{(s^2+1)(s-3)}.$$

(9 marks)

(iii) The variables $x_1(t)$ and $x_2(t)$ satisfy the coupled system of ordinary differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 2x_1 - \frac{1}{2}x_2 + e^t\cosh(2t),$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -4x_1 + 3x_2,$$

subject to the initial conditions $x_1(0) = 1$ and $x_2(0) = 2$. Using Laplace transforms, solve for $x_1(t)$. Hence find $x_2(t)$ without using its Laplace transform. ***(11 marks)***

4 The temperature T on a rectangular metal sheet satisfies the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

where $0 \leq x \leq a$ and $0 \leq y \leq b$, with a and b given constants. The temperature at $x = 0$, $x = a$ and $y = 0$ is fixed at 0 degree.

(i) Using the *method of separation of variables*, show that the general solution can be written as

$$T(x, y) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi\, y}{a} \sin \frac{n\pi\, x}{a}.$$

where $b_n (n = 1, 2, ...)$ are constant coefficients. ***(19 marks)***

(ii) If the temperature on the boundary $y = b$ is held at

$$T(x, b) = 2,$$

determine the coefficients $b_n (n = 1, 2, ...)$. ***(6 marks)***

5 (i) Evaluate the integral

$$\int_0^1 \int_y^{y^3} e^{\frac{2x}{y}}\, \mathrm{d}x\, \mathrm{d}y\,.$$

(6 marks)

(ii) Evaluating the following integral by first changing the order of integration:

$$\int_0^2 \int_{x^2}^4 x e^{y^2} \mathrm{d}y\, \mathrm{d}x\,.$$

(7 marks)

(iii) *Using a change of coordinates*, evaluate the integral

$$\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} \frac{x\ \mathrm{d}y\, \mathrm{d}x}{[x^2 + y^2]^2}\,.$$

(12 marks)

End of Question Paper

For use with MAS253 first semester examination

Formulae for use in L2 Mechanical Engineering Mathematics Examination

These results may be quoted without proof unless proofs are asked for in the question.

Trigonometry

$$\sin 2P = 2\sin P \cos P,$$

$$\cos 2P = \cos^2 P - \sin^2 P = 2\cos^2 P - 1 = 1 - 2\sin^2 P,$$

$$\cos P \cos Q = \frac{1}{2}\{\cos(P+Q) + \cos(P-Q)\},$$

$$\sin P \sin Q = -\frac{1}{2}\{\cos(P+Q) - \cos(P-Q)\},$$

$$\sin P \cos Q = \frac{1}{2}\{\sin(P+Q) + \sin(P-Q)\}.$$

Geometric progression

The partial sum to n terms of

$$a + ar + ar^2 + \ldots\ldots + ar^{n-1} + \ldots\ldots$$

is

$$S_n = a(1-r^n)/(1-r), \quad r \neq 1 \quad .$$

Taylor Series for functions of one variable (x)

The Taylor series of $f(x)$ about $x=a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maclaurin series of $f(x)$ is the special case of the Taylor series when $a=0$:

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!} f''(0)\,x^2 + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Important examples of Maclaurin series are:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \ldots \qquad (R \text{ is infinite})$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots. \qquad (R \text{ is infinite})$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots. \qquad (R \text{ is infinite})$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots \qquad (R = 1)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots. \qquad (R = 1)$$

R is the radius of convergence.

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots..$$

If n is positive and integer, series terminates.

If n is negative or non-integer (or both), the series is an infinite series with the radius of convergence, $R = 1$.

Fourier Series

The Fourier series of $f(x)$ for $-l < x < l$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where $$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \quad ,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \quad n = 1, 2, \ldots$$

Laplace Transform

The Laplace Transform of $f(t)$ is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt \quad .$$

For special cases, see later page.

Partial Differentiation

$$\delta F = F(x+\delta, y+\varepsilon) - F(x,y) \cong \delta \frac{\partial F}{\partial x} + \varepsilon \frac{\partial F}{\partial y}$$

Chain Rules:

1. Suppose that $F = F(x,y)$ and that x and y are functions of t, i.e. $x = x(t), y = y(t)$, then

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \quad .$$

2. Suppose that $F = F(x,y)$ and that x and y are functions of the variables u and v, i.e. $x = x(u,v), y = y(u,v)$, then

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}; \qquad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} \quad .$$

Taylor Series for functions of two variables (x, y)

The Taylor series of $f(x,y)$ about $x = a, y = b$ is

$$\begin{aligned} f(x,y) &= f(a,b) + \{(x-a)\, f_x(a,b) + (y-b)\, f_y(a,b)\} + \\ &+ \frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + \\ &+ (y-b)^2 f_{yy}(a,b)\} + \\ &+ \dots \end{aligned}$$

Here $f_x = \frac{\partial f}{\partial x}$ etc.

Partial Differential Equations (2 independent variables)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$ Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{K}\frac{\partial V}{\partial t}$$ Heat conduction (or diffusion) eqn.

equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$ Wave equation

General Solution of ODEs

$$X'' = -\omega^2 X \Rightarrow X(x) = A\cos \omega x + B\sin \omega x$$

$$X'' = \omega^2 X \Rightarrow X(x) = C\cosh \omega x + D\sinh \omega x$$

$$\text{or } Ee^{\omega x} + Fe^{-\omega x}$$

$$T' = kT \Rightarrow T(t) = Ae^{kt}$$

Table of Laplace Transforms	
$f(t)$	$F(s) = L(f(t))$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{iv}(t)$	$s^4 F(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$
1	$1/s$
t	$1/s^2$
$t^{n-1}/(n-1)!\ (n \geq 1)$	$1/s^n$
e^{at}	$\frac{1}{s-a}$
$\frac{1}{a}\sin at$	$\frac{1}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\frac{1}{a}\sinh at$	$\frac{1}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{\sin at - at\cos at}{2a^3}$	$\frac{1}{(s^2+a^2)^2}$
$\frac{t\sin at}{2a}$	$\frac{s}{(s^2+a^2)^2}$
$e^{at} f(t)$	$F(s-a)$, where $F(s) = L(f(t))$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$u(t-a)$	e^{-as}/s
$u(t-a)\ f(t-a)$	$e^{-as} F(s)$, where $F(s) = L(f(t))$