The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2016–2017

Continuum Mechanics 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Write down in full the following expressions:

$$\text{(a)}\ x_i y_i \quad \text{(b)}\ t_i = T_{ij} n_j \quad \text{(c)}\ \delta_{jk} T_{ki} n_k .$$

(6 marks)

(ii) You are given that the components of tensor $\mathbf{T}$ in the old and new Cartesian coordinates are related by $T'_{ij} = a_{ik} a_{jl} T_{kl}$, where a_{ij} are the entries of the trasformation matrix $\hat{\mathbf{A}}$ from the old to the new coordinates. Show that the matrices $\hat{\mathbf{T}}'$ with the entries T'_{ij} and $\hat{\mathbf{T}}$ with the entries T_{ij} are related by

$$\hat{\mathbf{T}}' = \hat{\mathbf{A}} \hat{\mathbf{T}} \hat{\mathbf{A}}^T ,$$

where the superscript T indicates the transposition. ***(4 marks)***

(iii) You are given that tensor $\mathbf{T}$ is invariant under the coordinate rotation about the x_3 axis defined by the transformation matrix

$$\hat{\mathbf{A}} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ϕ is an arbitrary rotation angle. Show that the matrix of components of tensor $\mathbf{T}$ must have the form

$$\hat{\mathbf{T}} = \begin{pmatrix} T_1 & -T_2 & 0 \\ T_2 & T_1 & 0 \\ 0 & 0 & T_3 \end{pmatrix}, \qquad (*)$$

where T_1, T_2, and T_3 are arbitrary quantities. Show that the condition $(*)$ is necessary and sufficient. ***(15 marks)***

2 (i) Give the definition of a streamline. Show that, in Cartesian coordinates x_1, x_2, x_3, the system of equations for streamlines can be written in the form

$$\frac{dx_1}{d\lambda} = v_1(\boldsymbol{x}, t), \quad \frac{dx_2}{d\lambda} = v_2(\boldsymbol{x}, t), \quad \frac{dx_3}{d\lambda} = v_3(\boldsymbol{x}, t),$$

where $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity, $\boldsymbol{x} = (x_1, x_2, x_3)$, and λ is the parameter in the parametric equation of a streamline. ***(9 marks)***

(ii) The velocity field of a planar motion with circulation generated by a point source with the constant intensity $Q > 0$ is given by

$$\boldsymbol{v} = \frac{Qx_1 - \Gamma x_2}{x_1^2 + x_2^2}\,\boldsymbol{e}_1 + \frac{\Gamma x_1 + Qx_2}{x_1^2 + x_2^2}\,\boldsymbol{e}_2, \qquad (*)$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are the base vectors, and $2\pi\Gamma$ is the constant velocity circulation.

(a) Use the variable substitution

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi$$

to obtain the system of equations

$$\frac{dr}{d\lambda} = \frac{Q}{r}, \quad \frac{d\phi}{d\lambda} = \frac{\Gamma}{r^2}$$

defining streamlines in the x_1x_2-plane in polar coordinates r, ϕ for the velocity field $(*)$. ***(6 marks)***

(b) For the velocity field $(*)$, find the equation of a streamline that contains the point of the x_1x_2-plane with polar coordinates (r_0, ϕ_0). Write this equation in the form $r = r(\phi)$. ***(10 marks)***

3 (i) Give the definitions of the principal directions and principal stresses of the stress tensor $\mathbf{T}$, and write down the equation defining the principal directions and stresses. ***(5 marks)***

(ii) Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be the principal directions of the stress tensor $\mathbf{T}$ corresponding to the principal stresses T_1, T_2 and T_3. Show that in Cartesian coordinates with the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ the matrix $\hat{\mathbf{T}}$ of the tensor $\mathbf{T}$ has the diagonal form with T_1, T_2 and T_3 on the main diagonal. ***(5 marks)***

(iii) Give the expression for the surface traction $\mathbf{t}$ in terms of the stress tensor and the unit normal vector $\mathbf{n}$ to the surface. ***(3 marks)***

(iv) Consider the surfaces with the normal unit vectors

$$\boldsymbol{n}_1 = \frac{1}{\sqrt{6}}\,\boldsymbol{e}_1 + \frac{1}{\sqrt{3}}\,\boldsymbol{e}_2 + \frac{1}{\sqrt{2}}\,\boldsymbol{e}_3,$$

$$\boldsymbol{n}_2 = \frac{1}{\sqrt{3}}(\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3),$$

$$\boldsymbol{n}_3 = \frac{1}{\sqrt{2}}\,\boldsymbol{e}_1 + \frac{1}{2}(\boldsymbol{e}_2 - \boldsymbol{e}_3),$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the unit vectors in the principal directions of $\mathbf{T}$. Denote the corresponding surface tractions by $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$ respectively. You are given that $\boldsymbol{t}_2$ is perpendicular to $\boldsymbol{n}_2$, $\boldsymbol{t}_3$ is perpendicular to $\boldsymbol{n}_1$, $|\boldsymbol{t}_2| = 2\sqrt{2}\ \mathrm{N/m^2}$, and $T_2 > 0$. Calculate T_1, T_2, and T_3. ***(12 marks)***

4 (i) Using Euler's equation for incompressible homogeneous fluid written in the Gromeka-Lamb form,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\|\boldsymbol{v}\|^2 + \varphi\right),$$

where φ is the body force potential, derive Bernoulli's integral for fluid stationary motion,

$$p + \frac{\rho}{2}\|\boldsymbol{v}\|^2 + \rho\varphi = \text{const}$$

along a streamline. ***(8 marks)***

(ii) A tank in the form of a cube with the side $l = 1$ m is open from above and filled with water up to the top. There is a circular hole of radius r at the bottom of the tank, so that water leaks through this hole.

(a) Considering the flow as approximately stationary and water as an ideal incompressible fluid, show that Bernoulli's integral takes the form

$$v^2 = 2gh,$$

where $v = \|\boldsymbol{v}\|$ and h is the water depth in the tank. ***(5 marks)***

(b) Show that h satisfies the equation

$$\frac{dh}{dt} = -\frac{\pi r^2 \sqrt{2gh}}{l^2}$$

(5 marks)

(c) You are given that, after the time interval $T = 10$ min., only one half of the initial volume of water remains in the tank. Obtain the expression for r in terms of l and g, the acceleration due to gravity. Then use this expression to calculate the numerical value of r. [You may take $g = 10\,\mathrm{m\,s^{-2}}$.] ***(7 marks)***

5 (i) You are given that, in linear elasticity, the equilibrium equation is given by

$$\nabla \cdot \mathbf{T} + \rho_0 \boldsymbol{b} = 0,$$

where $\boldsymbol{b}$ is the body force and ρ_0 is the density. You are also given that, in an isotropic material, the stress tensor has the from

$$\mathbf{T} = \lambda \mathbf{I} \nabla \cdot \boldsymbol{u} + \mu \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right], \qquad (*)$$

where $\boldsymbol{u}$ is the displacement, $\mathbf{I}$ is the unit tensor, and λ and μ are the Lamé constants. Show that, when $\boldsymbol{b} = 0$, the equilibrium equation reduces to

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} = 0. \qquad (\dagger)$$

(6 marks)

(ii) There is a cylinder of radius R and height H made of elastic isotropic material. The cylinder is put on a horizontal surface. The surface is smooth, so that the base of the cylinder can slide without friction with respect to the horizontal surface. The deformation of the surface can be neglected. There is the force of magnitude F appied to the flat top of the cylinder. The force is directed donward and evenly distributed, meaning that the pressure is the same at all points of the flat top.

(a) You can assume that, in cylindrical coordinates r, ϕ, z with the z-axis directed upward and coinsiding with the cylinder axis, the displacement in the cylinder has the form $\boldsymbol{u} = u_r(r)\boldsymbol{e}_r + u_z(z)\boldsymbol{e}_z$, where $\boldsymbol{e}_r$ and $\boldsymbol{e}_z$ are the unit vectors in the r and z-direction. Show that, for this particular form of $\boldsymbol{u}$, equation ($\dagger$) reduces to the system of two equations,

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r u_r)}{\partial r}\right) = 0, \qquad \frac{\partial^2 u_z}{\partial z^2} = 0.$$

Assuming that the origin of cylindrical coordinates is on the horizontal plane, show that the general solution to this system of equations regular at $r = 0$ is given by $u_r = Ar$, $u_z = Bz$, where A and B are constant.

[You can use without proof that, for $\boldsymbol{u} = u_r(r)\boldsymbol{e}_r + u_z(z)\boldsymbol{e}_z$, $\nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u})$. The general expression for the divergence in cylindrical coordinates is $\nabla \cdot \boldsymbol{v} = \dfrac{1}{r}\dfrac{\partial (r v_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{\partial v_z}{\partial z}$.]

(8 marks)

 Question 5 continued on next page

5 (continued)

(b) Using equation ($*$) show that

$$\mathbf{T} = [2A(\lambda + \mu) + \lambda B](\boldsymbol{e}_r\boldsymbol{e}_r + \boldsymbol{e}_\phi\boldsymbol{e}_\phi) + [2A\lambda + B(\lambda + 2\mu)]\boldsymbol{e}_z\boldsymbol{e}_z,$$

where $\boldsymbol{e}_\phi$ is the unit vector in the ϕ-direction. Use this expression to calculate the surface traction at the flat top of the cylinder and at its side surface. Then, using the condition of continuity of the surface traction at these two surfaces and neglecting the air pressure express the constants A and B in terms of R, H, F, and the Lamé constants λ and μ.

[You can use without proof that, for $\boldsymbol{u} = u_r(r)\boldsymbol{e}_r + u_z(z)\boldsymbol{e}_z$,
$\nabla\boldsymbol{u} = \dfrac{\partial u_r}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_r + \dfrac{u_r}{r}\boldsymbol{e}_\phi\boldsymbol{e}_\phi + \dfrac{\partial u_z}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_z.$] ***(11 marks)***

End of Question Paper