The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016–17**

Introduction to Relativity **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) (a) State the two postulates of special relativity. ***(4 marks)***

(b) Describe what is meant by "the relativity of simultaneity". ***(3 marks)***

(ii) The inertial frame $\tilde{R} : (c\tilde{t}, \tilde{x})$ is moving at a constant velocity v relative to the inertial frame $R : (ct, x)$, such that the two frames are related by the two-dimensional *Lorentz transformation*

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \qquad \gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

A rod of length L is at rest in frame $\tilde{R}$, with its ends at $\tilde{x} = 0$ and $\tilde{x} = L$.

(a) An observer in R measures both ends of the rod simultaneously.
In R, the measurement events have coordinates $(0, 0)$ and $(0, \ell)$.
In $\tilde{R}$, the measurement events have coordinates $(0, 0)$ and (cT, L).
Use the Lorentz transformation to find T and ℓ. ***(4 marks)***

(b) Are the measurement events simultaneous in $\tilde{R}$? If not, which is first? ***(2 marks)***

(c) A rod of length 4 metres in $\tilde{R}$ is travelling with velocity $v = \sqrt{3}c/2$ in R. How long is the rod in R? ***(3 marks)***

(d) The right-hand end of the rod passes into a room of width 3 metres which is stationary in R, and it collides with the right-hand wall.
At the moment of collision with the wall, is the rod entirely within the room? Describe the situation in both R and $\tilde{R}$. ***(4 marks)***

(e) Draw a *spacetime diagram* to illustrate (ii)(d). Mark three events:
A, the collision between the right-hand end of the rod and the wall;
B, the left-hand end of the rod simultaneous with A in R;
C, the left-hand end of the rod simultaneous with A in $\tilde{R}$.
(5 marks)

2 A Lorentz transformation L is represented by a 4×4 matrix satisfying

$$L^T g L = g, \qquad \text{where} \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

with L^T the matrix transpose.

(i) Show that if L represents a Lorentz transformation then:

(a) L^n is also a Lorentz transformation, for any positive integer n (here L^n denotes L multiplied by itself n times). ***(3 marks)***

(b) The inverse L^{-1} is also a Lorentz transformation. ***(4 marks)***

(c) The transpose L^T is also a Lorentz transformation. ***(4 marks)***

(ii) Let

$$L = \frac{1}{2}\begin{pmatrix} \sqrt{8} & \sqrt{2} & -\sqrt{2} & 0 \\ 2 & 2 & -2 & 0 \\ 0 & 1 & 1 & \sqrt{2} \\ 0 & 1 & 1 & -\sqrt{2} \end{pmatrix}.$$

Show that L is an *improper orthochronous Lorentz transformation.* ***(14 marks)***

3 (i) An astronaut is travelling at uniform speed $u = c/2$ in the positive-x direction of an inertial frame R. The astronaut's rest frame is $\tilde{R}$.

(a) The astronaut fires a projectile at speed $v = c/2$ in the positive-$\tilde{x}$ direction of her frame $\tilde{R}$. Find the projectile's speed in R.

(4 marks)

(b) The astronaut fires a projectile at speed $v = c/\sqrt{3}$ in the positive-$\tilde{y}$ direction of her frame $\tilde{R}$. By applying a standard inverse Lorentz transformation to the four-velocity $\tilde{V} = (c\gamma_v, 0, v\gamma_v, 0)$, where $\gamma_v \equiv (1 - v^2/c^2)^{-1/2}$, find:

- the projectile's speed w in R.

- the angle of its trajectory θ relative to the x-axis of R.

(6 marks)

(ii) Two particles A and B are travelling at uniform three-velocities $\mathbf{u}$ and $\mathbf{v}$, respectively, in an inertial frame R. Their four-velocities are

$$U = (\gamma_u c, \gamma_u \mathbf{u}), \qquad V = (\gamma_v c, \gamma_v \mathbf{v}).$$

(a) Define the Lorentz bracket.
Show that $g(U, U) = c^2$. Classify U as timelike, spacelike or null.

(4 marks)

(b) Let $\mathbf{w}$ denote the velocity of particle A in particle B's rest frame. Using the frame-invariance of $g(U, V) = c^2\gamma_u\gamma_v(1 - \mathbf{u}\cdot\mathbf{v}/c^2)$, or otherwise, show that its speed $w = |\mathbf{w}|$ is given by

$$w^2 = \frac{|\mathbf{v} - \mathbf{u}|^2 - |\mathbf{u} \times \mathbf{v}|^2 / c^2}{(1 - \mathbf{u}\cdot\mathbf{v}/c^2)^2}.$$

(11 marks)

(You may use the vector identity $|\mathbf{u} \times \mathbf{v}|^2 = u^2v^2 - (\mathbf{u}\cdot\mathbf{v})^2$ without proof. Here $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$ denote the standard scalar product and vector product, respectively.)

4 (i) Consider an inertial observer with four-velocity V, and two events A and B which are connected by a displacement four-vector X. Show that the two events are simultaneous according to the inertial observer if and only if

$$g(X, V) = 0.$$

(5 marks)

(ii) An astronaut moving in an inertial frame R has a displacement vector $X(\tau)$ relative to the event O at the origin of the frame, where

$$X(\tau) = \left(\frac{c^2}{a}\sinh\rho, \frac{c^2}{a}\cosh\rho, 0, 0\right), \qquad \rho(\tau) = \frac{a\tau}{c}.$$

Here a is constant and τ is the astronaut's proper time.

(a) Find the four-velocity $V(\tau) = \dfrac{dX}{d\tau}$.
Find the four-acceleration $A(\tau)$. ***(4 marks)***

(b) Find the speed of the astronaut in frame R.
(Give your answer in terms of a, c and τ). ***(2 marks)***

(c) Show that $g(V, A) = 0$.
Find $g(A, A)$. ***(3 marks)***

(iii) (a) Show that event O at the origin of R is simultaneous with *every* event on the astronaut's world-line, according to the astronaut.
Find the distance to O according to the astronaut. ***(6 marks)***

(b) Sketch the world-line of the astronaut on a 2D spacetime diagram. Indicate the set of all events that *can not* send a future-pointing light ray to the astronaut, by shading a certain region of the spacetime diagram. ***(5 marks)***

5 (i) (a) Define the *rest mass* m and *four-momentum* P of a particle.

(2 marks)

(b) A particle of rest mass m_1 is travelling at speed u in the positive x-direction of an inertial frame. It collides with a stationary particle of rest mass m_2, to form a composite particle of rest mass M travelling at speed v in the same direction. Using the conservation of four-momentum and/or the Lorentz bracket, show that:

$$\begin{aligned} M^2 &= (m_1 + m_2)^2 + 2m_1 m_2(\gamma_u - 1), \\ v &= \frac{m_1\gamma_u}{m_1\gamma_u + m_2}\, u. \end{aligned}$$

(11 marks)

(ii) The 'Breakthrough Starshot' initiative proposes to accelerate a spacecraft of rest mass 1 gram to 20% of the speed of light, by shining a high-powered laser on to a solar sail.

(a) By considering the Lorentz bracket of the four-momentum $P = (E/c, \mathbf{p})$, or otherwise, derive the dispersion relation

$$E^2 = p^2c^2 + m^2c^4.$$

(3 marks)

(b) Suppose a laser shines with a power of 100 GW onto a solar sail. What is the rate of momentum transfer? ***(3 marks)***

(c) Estimate how long it would take to accelerate a 1g spacecraft to 20% of the speed of light. ***(3 marks)***

(d) Calculate the kinetic energy of the spacecraft. Express your answer as a proportion of the energy in the first nuclear bomb ($E \approx 63 \times 10^{12}$ J). ***(3 marks)***

End of Question Paper