The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2016–17

WAVES

2 hours

Marks will be awarded for your best FOUR *answers. The marks awarded to each question or section of question are shown in italics.*

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1 A string with uniform mass per unit length ρ is under a tension F, and undergoes small transverse vibrations. Let the displacement of the point at distance x along the string at time t be $y(x,t)$.

(i) Show that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

to good approximation, where the constant $c^2 = F/\rho$.

(5 marks)

(ii) Derive the general solution (*d'Alembert's solution*) of this equation.

(11 marks)

(iii) In a particular case the string is unbounded in both directions, i.e. $-\infty < x < \infty$. At $t = 0$

$$y(x,0) = 0, \quad \dot{y}(x,0) = V\cos(2kx),$$

where V and k are constants. Find $y(x,t)$ and all values of x for which $y(x,t) = 0$. Describe the motion very briefly.

(9 marks)

2 A uniform finite string of length l and mass per unit length ρ occupies the interval $0 \le x \le l$ and undergoes transverse vibrations with displacement $y(x,t)$, where $c^2 y_{xx} = y_{tt}$, and c^2 is a constant. The tension in the string is ρc^2. You are given that

(a) $y(0,t) = y(l,t) = 0$;
(b) $y(x,0) = (h/l^2)x(l-x)$ where h is a constant;
(c) $\dot{y}(x,0) = 0$;
(d) $y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$, where the $a_n\,(n = 1,2,3,..)$ are constants.

(i) Verify that the series in (d) satisfies the PDE and conditions (a) and (c).

(8 marks)

(ii) Find the a_n so that (b) is satisfied.

(9 marks)

(iii) Deduce that the potential energy stored in the string at time t is

$$\frac{16\rho h^2 c^2}{\pi^4 l}\sum_{m=0}^{\infty}\frac{\cos^2\left\{\dfrac{(2m+1)\pi ct}{l}\right\}}{(2m+1)^4}.$$

(8 marks)

3 In a compressible, static and uniform gas, with constant density ρ_0 and pressure p_0, due to the passage of a sound disturbance, there are *small* changes in density ρ and pressure p.

(i) Given that the exact equation of continuity is

$$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0,$$

where $\mathbf{u}(\mathbf{x}, t) = u(\mathbf{x}, t)\mathbf{i} + v(\mathbf{x}, t)\mathbf{j} + w(\mathbf{x}, t)\mathbf{k}$ is the velocity field of the perturbed state, obtain a valid approximation to the exact continuity equation in the limit of linear theory (i.e. small changes).

(5 marks)

(ii) Using Newton's Second Law, again in linear approximation, and given that p is a function of ρ, show that

$$\rho_{tt} = c^2(\rho_{xx} + \rho_{yy} + \rho_{zz}),$$

where c^2 is a constant which should be defined.

(10 marks)

(iii) In a particular case $\rho = \rho_0(1 + s)$ and supposing a potential flow (i.e. $u = \phi_x$, $v = \phi_y$ and $w = \phi_z$, where ϕ is the velocity potential) show that for the linearised continuity equation

$$s_t = -(\phi_{xx} + \phi_{yy} + \phi_{zz}).$$

Provided that all disturbances decay to a steady state as $\mid \mathbf{x} \mid \to \infty$, deduce that

$$c^2 s = -\phi_t,$$

and hence show that

$$\phi_{tt} = c^2(\phi_{xx} + \phi_{yy} + \phi_{zz}),$$

i.e. the velocity potential also satisfies the three-dimensional form of the wave equation.

(10 marks)

4 The equilibrium position of the free surface of a liquid of infinite depth is $z = 0$, where z is measured vertically upwards. A surface wave causes the displacement of this surface to be $\eta(x, t)$, where x is measured along the undisturbed surface and

$$\eta = a\cos(kx - \omega t),$$

with a, k and ω being positive constants with a small. You are given that the velocity potential $\phi = \phi(x, z, t)$ satisfies

$$\phi_{xx} + \phi_{zz} = 0.$$

You are also given that (a) $\phi_z \to 0$ as $z \to -\infty$; (b) $\phi_z = \eta_t$ at $z = 0$; (c) $\phi_t + g\eta = 0$ at $z = 0$.

(i) Give a brief physical interpretation of (a), (b) and (c).

(6 marks)

(ii) Find $\phi(x, z, t)$ and show that $\omega^2 = gk$.

(13 marks)

(iii) Determine the phase velocity c and the group velocity c_g in terms of k. Show that $c_g = c/2$, and state what is the speed of the propagation of energy.

(6 marks)

5 (i) Solve, using the method of characteristics, the equation

$$yz_x + xz_y = xy,$$

given that $z = e^{-y^2}$ on $x = 0$ for $y \geq 0$ and that $z = e^{-x^2}$ on $y = 0$ for $x \geq 0$. $\left[\textit{We use the notation } z_x = \dfrac{\partial z}{\partial x} \textit{ etc.}\right]$

(13 marks)

(ii) On what region D in the x–y plane is the solution defined uniquely? Verify that z is everywhere continuous in D, but that z_x and z_y are discontinuous across one curve. Determine the curve of discontinuity.

(7 marks)

(iii) More generally, suppose that within a region D^* in the x–y plane, the solution $z = z(x, y)$ of

$$Pz_x + Qz_y = R,$$

where P, Q, R are continuous functions of x, y, z, is everywhere continuous, but that there may be discontinuities in z_x and z_y across a curve Γ. Show that $dy/dx = Q/P$ on Γ.

(5 marks)

End of Question Paper