The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016–17**

Topics in Number Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please read the questions carefully. Your solutions should be written legibly and give enough details to make it clear how you arrived at your answers. Usage of calculators is not allowed.

1 ***(20 marks)***

(i) Define the Euler function $\phi(n)$. ***(2 marks)***

(ii) State the Möbius Inversion Formula. ***(2 marks)***

(iii) Define a primitive root modulo n. ***(2 marks)***

(iv) State the Gauss reciprocity law (both parts). ***(4 marks)***

(v) Define a primitive Pythagorean triple. ***(2 marks)***

(vi) Give the condition for existence of a solution of $m = x^2 + y^2$ in terms of the prime factorization of m. ***(2 marks)***

(vii) Define the Mersenne number M_n and state the theorem about divisors of M_p. ***(2 marks)***

(viii) Define a perfect number and state the theorem about even perfect numbers. ***(2 marks)***

(ix) State the theorem about solutions of Pell's equation in terms of continued fractions. ***(2 marks)***

2 ***(30 marks)***

(i) What is the remainder when $2015^{2016\times 2017\times 2018}$ is divided by 11?
(5 marks)

(ii) Use the prime factorization of $n = 150$ to compute $\tau(n)$, $\sigma(n)$, $\mu(n)$, $\phi(n)$.
(5 marks)

(iii) Find the orders of $\overline{3}$ and $\overline{4}$ in $\mathbb{Z}_{17}^{\star}$. ***(5 marks)***

(iv) Describe explicitly all primes p for which the equation $x^2 + 4x + 1 \equiv 0 \pmod{p}$ has two solutions. You do not need to find the solutions.
(5 marks)

(v) Find *all* Pythagorean triples, *not necessarily primitive*, of the form $28, y, z$ $(y, z > 0)$. ***(5 marks)***

(vi) Find the continued fraction representation of $\sqrt{6}$ and compute its convergents C_0, C_1, C_2, C_3, C_4. ***(5 marks)***

3 *(30 marks)*

(i) Prove the following property of the τ function:

$$\sum_{d|n} \tau^3(d) = \left(\sum_{d|n} \tau(d) \right)^2 .$$

(6 marks)

(ii) For each of the statements below, if it is true, give a proof, and if it is false, give a counterexample:

(a) If a is a primitive root modulo p, then a is a quadratic nonresidue modulo p.

(b) If a is quadratic nonresidue modulo p, then a is a primitive root modulo p.

(6 marks)

(iii) Prove the formula for solutions $u, v \in \mathbb{Q}$ of $u^2 + v^2 = 1$ in terms of one parameter $s \in \mathbb{Q} \cup \{\infty\}$. *(6 marks)*

(iv) Let $c(n)$ be the number of ways to represent a sum of n pounds using the 1 and 2 pound coins and 5 pound notes. For instance, $c(5) = 4$ corresponding to

$$5 = 1 + 1 + 1 + 1 + 1 = 2 + 1 + 1 + 1 = 2 + 2 + 1 = 5.$$

Give a formula for the generating function for $c(n)$, and *using it* compute $c(1), c(2), \ldots, c(6)$. *(6 marks)*

(v) We know that any real number $\alpha \in \mathbb{R}$ can be represented by a continued fraction. Explain whether such a representation is unique. [Hint: The answer to this question depends on whether α is a rational number or not; explore both cases.] *(6 marks)*

4 ***(20 marks)***

(i) (a) Prove the following theorem of Wilson:

$$(p-1)! \equiv -1 \pmod p$$

(4 marks)

(b) Using Wilson's Theorem prove that

$$1^2 \cdot 3^2 \cdot 5^2 \cdot \cdots \cdot (p-2)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod p\,.$$

(3 marks)

(ii) In this question we study the equation

$$m = x^2 + 3y^2,\ x, y \in \mathbb{Z}.$$

First describe which primes p can be written in the form $x^2 + 3y^2$. Then formulate and prove the condition on the natural number m for the solution of $m = x^2 + 3y^2$ to exist. ***(7 marks)***

(iii) Let u_n denote the Fibonacci sequence: $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$. In this question we consider the remainders $u_n \pmod p$ for a prime p.

Assume that $p \equiv \pm 1 \pmod 5$. Prove that there are elements α and β in $\mathbb{Z}_p$ for which there is an analogue of Binet's formula for $u_n \pmod p$. Deduce that $u_n \pmod p$ is periodic with period $p-1$:

$$u_{n+p-1} \equiv u_n \pmod p\,.$$

(6 marks)

End of Question Paper