SCHOOL OF MATHEMATICS AND STATISTICS  
Autumn Semester 2016–17

MAS331 Metric Spaces  
2 hours 30 minutes

Answer four questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

Please leave this exam paper on your desk  
Do not remove it from the hall

Registration number from U-Card (9 digits)  
to be completed by student

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The following is an incorrect definition of a metric $d$, defined on a set $X$. Identify each place where the definition is wrong, and replace this with a corrected version. (Just writing out the correct definition of a metric space, without any commentary, is not an acceptable solution.):

“A metric is a mapping $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) > 0$ for all $x, y \in X$ and if $x = y$, then $d(x, y) = 0$,
2. $d(x, y) + d(y, x) = 0$, for all $x, y \in X$,
3. $d(x, z) - d(y, z) - d(x, y) \geq 0$, for all $x, y, z \in X$.”

Let $(X, d)$ be a metric space. Derive the inequality

$$|d(x, z) - d(y, z)| \leq d(x, y),$$

for all $x, y, z \in X$. (6 marks)

(iii) (a) Define $d : C[0, 1] \times C[0, 1] \to \mathbb{R}$ where

$$d(f, g) = \int_0^1 |f(x) - g(x)| \rho(x) dx,$$

where $\rho \in C[0, 1]$ satisfies $\rho(x) > 0$ for all $x \in [0, 1]$. Prove that $d$ is a metric. [Hint: You may use the fact that if $h : [a, b] \to \mathbb{R}$ is a continuous function for which $h(x) \geq 0$ for all $a \leq x \leq b$ then

$$\int_a^b h(x) dx = 0 \text{ implies that } h(x) = 0 \text{ for all } a \leq x \leq b.]$$

(b) For each $r > 0$, consider the metric $d_r$ on $C[0, 1]$ obtained by taking $\rho(x) = \frac{1}{2^r(1 + x)^r}$, for each $x \in [0, 1]$ in (iii)(a).

(α) Calculate $d_r(f, g)$ where $f(x) = 1 + x^2$ and $g(x) = 2 + x^2$. (2 marks)

(β) Is $d_\infty$ a metric on $C[0, 1]$, where $d_\infty(f, g) = \lim_{r \to \infty} d_r(f, g)$? Give some evidence to support your conclusion. (3 marks)
Let \((X, d)\) be a metric space.

(i) (a) Explain what it means for a subset of \(X\) to be (I) open, (II) closed.  
\hspace{1cm} (4 marks)

(b) Show that if \(A \subseteq X\) is open, then its complement, \(A^c := X \setminus A\) is closed.  
\hspace{1cm} (5 marks)

(c) Is the following statement correct? “Every subset of a metric space is either open or closed.” Give a reason for your answer.  
\hspace{1cm} (2 marks)

(ii) Equip \(\mathbb{R}^2\) with the usual Euclidean metric.

(a) Fix \(a, b > 0\). Show that the set of points \(A\) is closed, where
\[ A = \left\{ (x, y) \in \mathbb{R}^2; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}. \]
\hspace{1cm} (A is the set of all points that lie on an ellipse whose shape is determined by \(a\) and \(b\).)  
\hspace{1cm} (4 marks)

(b) Is the set
\[ \hat{A} = \left\{ (x, y) \in \mathbb{R}^2; \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}, \]
open, closed or neither? Give a brief explanation for your answer.  
\hspace{1cm} (3 marks)

(iii) Consider the space \(C[0,1]\) with the supremum metric
\[ d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|. \]

(a) Write down an expression for the open ball \(B_r(0)\) of radius \(r > 0\), centred on the zero function \(g(x) = 0\) for all \(x \in [0,1]\).  
\hspace{1cm} (2 marks)

(b) Consider the function \(h(x) = 6x - x^2\) for all \(x \in [0,1]\). Is it true that \(h \in B_4(0)\), or that \(h \in B_6(0)\)? Give explicit calculations to support your answer.  
\hspace{1cm} (5 marks)
(i) Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces. Write down two (equivalent) definitions of what it means for a function \(f : X_1 \to X_2\) to be continuous. One of these should involve sequences, while the other uses \(\varepsilon\)'s and \(\delta\)'s.

\(4\) marks

(ii) Show that the following functions are continuous:

(a) \(f : (\mathbb{R}^2, d_2) \to (\mathbb{R}, d_1)\), where for all \(x, y \in \mathbb{R}\),

\[
f(x, y) = \cos(2x^2 - 3y).
\]

Here \(d_2\) is the Euclidean metric on \(\mathbb{R}^2\), and \(d_1\) is the standard metric on \(\mathbb{R}\).

\(4\) marks

(b) \(I : (C[0, 1], d_{\infty}) \to (\mathbb{R}, d_1)\), where

\[
I(f) = \int_0^1 f(x)dx,
\]

and \(d_{\infty}\) is the supremum metric, given by

\[
d_{\infty}(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.
\]

\(5\) marks

(iii) If \((X_1, d_1), (X_2, d_2)\) and \((X_3, d_3)\) are metric spaces, and \(f : X_2 \to X_3\) and \(g : X_1 \to X_2\) are continuous show that \(f \circ g\) is continuous from \(X_1\) to \(X_3\), where for all \(x \in X_1\),

\[
(f \circ g)(x) := f(g(x)).
\]

Hence show that the mapping \(f \to \left(\int_0^1 f(x)dx\right)^2\) is continuous from \((C[0,1], d_{\infty})\) to \((\mathbb{R}, d_1)\).

\(6\) marks

(iv) Explain what it means for a subset of a metric space to be compact. If \((X_1, d_1)\) and \((X_2, d_2)\) are metric spaces, \(A \subseteq X\) is compact, and \(f : X_1 \to X_2\) is continuous, show that \(f(A)\) is compact, where

\[
f(A) := \{y \in X_2; y = f(x) \text{ for some } x \in A\}.
\]

\(6\) marks
4 Let \((X, d)\) be a metric space.

\[\text{(i) } \begin{align*} &\text{(a) If } T : X \to X \text{ is a mapping, explain what it means for } T, \text{ (I) to have a fixed point, (II) to be a contraction.} \quad (2 \text{ marks}) \\
&\text{(b) Prove that every contraction is continuous.} \quad (4 \text{ marks}) \\
&\text{(c) State (without proof) the contraction mapping theorem.} \quad (2 \text{ marks}) \end{align*}\]

\[\text{(ii) } \text{Fix } x_1 \in X \text{ and let } f : X \to X \text{ be continuous. Define a sequence } (x_n) \text{ by } x_{n+1} = f(x_n) \text{ for } n \in \mathbb{N}. \text{ Show that if } (x_n) \text{ converges to some } x \in X, \text{ then } x \text{ is a fixed point of } f. \quad (3 \text{ marks})\]

\[\text{(iii) } \text{Fix } a \in \mathbb{R}. \text{ Show that a continuous function } f : [a, \infty) \to \mathbb{R}, \text{ which is } \text{differentiable on } (a, \infty), \text{ is a contraction if and only if there exists } 0 \leq k < 1 \text{ so that } |f'(x)| < k, \text{ for all } x > a. \text{ [Hint: Use the mean value theorem, which states that if } a \leq b < c, \text{ then there exists } y \in (b, c) \text{ so that } f(c) - f(b) = f'(y)(c - b).] \quad (6 \text{ marks}) \]

\[\text{(iv) } \text{Which of the following are contractions from } [a, \infty) \to \mathbb{R}? \]

\[\text{(a) } f(x) = 1 + e^{-2x}, \text{ with } a = 1, \quad (2 \text{ marks})\]

\[\text{(b) } f(x) = \frac{x - 3}{2x + 1}, \text{ with } a = 0.25. \quad (3 \text{ marks})\]

\[\text{(v) Define } k : [1, \infty) \to \mathbb{R} \text{ by } k(x) = 1 - x + e^{-2x}. \]

\[\text{What can you say about the set } k^{-1}(\{0\})? \quad (3 \text{ marks})\]
Let $(X, d)$ be a metric space.

(i) Explain what it means for
(a) A sequence $(x_n)$ in $X$ to be a Cauchy sequence. \hspace{1cm} (2 marks)
(b) The metric space to be complete. \hspace{1cm} (1 mark)

(ii) Show that a closed subspace of a complete metric space is itself complete
(with respect to the restricted metric). \hspace{1cm} (3 marks)

(iii) Let $a, b \in \mathbb{R}$, with $a < b$. Show that $A := \{f \in C[0,1]; a \leq f(x) \leq b \text{ for all } x \in [1/2, 3/4] \}$ is complete with respect to the $d_\infty$ metric. \hspace{1cm} (5 marks)

(iv) Recall that the metric $d_1$ on $C[-1,1]$ is given by
\[
d_1(f, g) = \int_{-1}^{1} |f(x) - g(x)| \, dx.
\]
For each $n \in \mathbb{N}$, let $f_n \in C[-1,1]$ be given by
\[
f_n(x) = \begin{cases} 
0 & \text{for } -1 \leq x \leq 0, \\
\frac{n}{x} & \text{for } 0 \leq x \leq \frac{1}{n}, \\
1 & \text{for } \frac{1}{n} \leq x \leq 1.
\end{cases}
\]
(a) Show that $d_1(f_n, f_m) = \frac{1}{2} \left| \frac{1}{n} - \frac{1}{m} \right|$. \hspace{1cm} (4 marks)
(b) Give a careful proof that $(f_n)$ is a Cauchy sequence. \hspace{1cm} (3 marks)
(c) Deduce that $(C[-1,1], d_1)$ is not a complete metric space. [Hint: You may use the fact that for $f \in C[-1,1]$ and $-1 \leq a < b \leq 1$, if $\int_a^b |f(t)| \, dt = 0$, then $f(t) = 0$ for all $a \leq t \leq b$.] \hspace{1cm} (7 marks)

End of Question Paper