The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2016–17

MAS331 Metric Spaces — 2 hours 30 minutes

*Answer* ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

**Please leave this exam paper on your desk**
**Do not remove it from the hall**

Registration number from U-Card (9 digits)
to be completed by student

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

Blank



Continued

**1** (i) The following is an incorrect definition of a metric $d$, defined on a set $X$. Identify each place where the definition is wrong, and replace this with a corrected version. (Just writing out the correct definition of a metric space, without any commentary, is not an acceptable solution.):

"A metric is a mapping $d : X \times X \to \mathbb{R}$ such that

(M1) $d(x, y) > 0$ for all $x, y \in X$ and if $x = y$, then $d(x, y) = 0$,

(M2) $d(x, y) + d(y, x) = 0$, for all $x, y \in X$,

(M3) $d(x, z) - d(y, z) - d(x, y) \geq 0$, for all $x, y, z \in X$." ***(5 marks)***

(ii) Let $(X, d)$ be a metric space. Derive the inequality

$$|d(x, z) - d(y, z)| \leq d(x, y),$$

for all $x, y, z \in X$. ***(6 marks)***

(iii) (a) Define $d : C[0, 1] \times C[0, 1] \to \mathbb{R}$ where

$$d(f, g) = \int_0^1 |f(x) - g(x)|\rho(x)dx,$$

where $\rho \in C[0, 1]$ satisfies $\rho(x) > 0$ for all $x \in [0, 1]$. Prove that $d$ is a metric. [Hint: You may use the fact that if $h : [a, b] \to \mathbb{R}$ is a continuous function for which $h(x) \geq 0$ for all $a \leq x \leq b$ then $\int_a^b h(x)dx = 0$ implies that $h(x) = 0$ for all $a \leq x \leq b$.]

***(9 marks)***

(b) For each $r > 0$, consider the metric $d_r$ on $C[0, 1]$ obtained by taking $\rho(x) = \frac{1}{2^r}(1 + x)^r$, for each $x \in [0, 1]$ in (iii)(a).

($\alpha$) Calculate $d_r(f, g)$ where $f(x) = 1 + x^2$ and $g(x) = 2 + x^2$. ***(2 marks)***

($\beta$) Is $d_\infty$ a metric on $C[0, 1]$, where $d_\infty(f, g) = \lim_{r \to \infty} d_r(f, g)$? Give some evidence to support your conclusion. ***(3 marks)***

**2** Let $(X, d)$ be a metric space.

(i) (a) Explain what it means for a subset of $X$ to be (I) *open*, (II) *closed*. ***(4 marks)***

(b) Show that if $A \subseteq X$ is open, then its complement, $A^c := X \setminus A$ is closed. ***(5 marks)***

(c) Is the following statement correct? "Every subset of a metric space is either open or closed." Give a reason for your answer. ***(2 marks)***

(ii) Equip $\mathbb{R}^2$ with the usual Euclidean metric.

(a) Fix $a, b > 0$. Show that the set of points $A$ is closed, where

$$A = \left\{ (x, y) \in \mathbb{R}^2; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}.$$

($A$ is the set of all points that lie on an ellipse whose shape is determined by $a$ and $b$.) ***(4 marks)***

(b) Is the set

$$\widehat{A} = \left\{ (x, y) \in \mathbb{R}^2; \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\},$$

open, closed or neither? Give a brief explanation for your answer. ***(3 marks)***

(iii) Consider the space $C[0, 1]$ with the supremum metric

$$d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

(a) Write down an expression for the open ball $B_r(0)$ of radius $r > 0$, centred on the zero function $g(x) = 0$ for all $x \in [0, 1]$. ***(2 marks)***

(b) Consider the function $h(x) = 6x - x^2$ for all $x \in [0, 1]$. Is it true that $h \in B_4(0)$, or that $h \in B_6(0)$? Give explicit calculations to support your answer. ***(5 marks)***

**3** (i) Let $(X_1, d_1)$ and $(X_2, d_2)$ be metric spaces. Write down two (equivalent) definitions of what it means for a function $f : X_1 \to X_2$ to be *continuous*. One of these should involve sequences, while the other uses $\epsilon$'s and $\delta$'s.

***(4 marks)***

(ii) Show that the following functions are continuous:

(a) $f : (\mathbb{R}^2, d_2) \to (\mathbb{R}, d_1)$, where for all $x, y \in \mathbb{R}$,

$$f(x, y) = \cos(2x^2 - 3y).$$

Here $d_2$ is the Euclidean metric on $\mathbb{R}^2$, and $d_1$ is the standard metric on $\mathbb{R}$.

***(4 marks)***

(b) $I : (C[0, 1], d_\infty) \to (\mathbb{R}, d_1)$, where

$$I(f) = \int_0^1 f(x)dx,$$

and $d_\infty$ is the supremum metric, given by

$$d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

***(5 marks)***

(iii) If $(X_1, d_1), (X_2, d_2)$ and $(X_3, d_3)$ are metric spaces, and $f : X_2 \to X_3$ and $g : X_1 \to X_2$ are continuous show that $f \circ g$ is continuous from $X_1$ to $X_3$, where for all $x \in X_1$,

$$(f \circ g)(x) := f(g(x)).$$

Hence show that the mapping $f \to \left( \int_0^1 f(x)dx \right)^2$ is continuous from $(C[0, 1], d_\infty)$ to $(\mathbb{R}, d_1)$.

***(6 marks)***

(iv) Explain what it means for a subset of a metric space to be *compact*. If $(X_1, d_1)$ and $(X_2, d_2)$ are metric spaces, $A \subseteq X$ is compact, and $f : X_1 \to X_2$ is continuous, show that $f(A)$ is compact, where

$$f(A) := \{y \in X_2; y = f(x) \text{ for some } x \in A\}.$$

***(6 marks)***

**4** Let $(X, d)$ be a metric space.

(i) (a) If $T : X \to X$ is a mapping, explain what it means for $T$, (I) to have a *fixed point*, (II) to be a *contraction*. ***(2 marks)***

(b) Prove that every contraction is continuous. ***(4 marks)***

(c) State (without proof) the *contraction mapping theorem*. ***(2 marks)***

(ii) Fix $x_1 \in X$ and let $f : X \to X$ be continuous. Define a sequence $(x_n)$ by $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. Show that if $(x_n)$ converges to some $x \in X$, then $x$ is a fixed point of $f$. ***(3 marks)***

(iii) Fix $a \in \mathbb{R}$. Show that a continuous function $f : [a, \infty) \to \mathbb{R}$, which is differentiable on $(a, \infty)$, is a contraction if and only if there exists $0 \leq k < 1$ so that $|f'(x)| < k$, for all $x > a$. [Hint: Use the mean value theorem, which states that if $a \leq b < c$, then there exists $y \in (b, c)$ so that $f(c) - f(b) = f'(y)(c - b)$.] ***(6 marks)***

(iv) Which of the following are contractions from $[a, \infty)$ to $\mathbb{R}$?

(a) $f(x) = 1 + e^{-2x}$, with $a = 1$, ***(2 marks)***

(b) $f(x) = \dfrac{x - 3}{2x + 1}$, with $a = 0.25$. ***(3 marks)***

(v) Define $k : [1, \infty) \to \mathbb{R}$ by

$$k(x) = 1 - x + e^{-2x}.$$

What can you say about the set $k^{-1}(\{0\})$? ***(3 marks)***

**5** Let $(X, d)$ be a metric space.

(i) Explain what it means for

(a) A sequence $(x_n)$ in $X$ to be a *Cauchy sequence*, ***(2 marks)***

(b) The metric space to be *complete*. ***(1 mark)***

(ii) Show that a closed subspace of a complete metric space is itself complete (with respect to the restricted metric). ***(3 marks)***

(iii) Let $a, b \in \mathbb{R}$, with $a < b$. Show that $A := \{f \in C[0,1]; a \leq f(x) \leq b \text{ for all } x \in [1/2, 3/4]\}$ is complete with respect to the $d_\infty$ metric. ***(5 marks)***

(iv) Recall that the metric $d_1$ on $C[-1,1]$ is given by

$$d_1(f,g) = \int_{-1}^{1} |f(x) - g(x)| \, \mathrm{d}x.$$

For each $n \in \mathbb{N}$, let $f_n \in C[-1,1]$ be given by

$$f_n(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq 0, \\ nx & \text{for } 0 \leq x \leq \dfrac{1}{n}, \\ 1 & \text{for } \dfrac{1}{n} \leq x \leq 1. \end{cases}$$

(a) Show that $d_1(f_n, f_m) = \dfrac{1}{2}\left|\dfrac{1}{n} - \dfrac{1}{m}\right|$. ***(4 marks)***

(b) Give a careful proof that $(f_n)$ is a Cauchy sequence. ***(3 marks)***

(c) Deduce that $(C[-1,1], d_1)$ is not a complete metric space. [Hint: You may use the fact that for $f \in C[-1,1]$ and $-1 \leq a < b \leq 1$, if $\displaystyle\int_a^b |f(t)| dt = 0$, then $f(t) = 0$ for all $a \leq t \leq b$.] ***(7 marks)***

## End of Question Paper