The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016-2017**

Complex Analysis **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) State, without proof, the triangle inequalities for $|\,z+w|$ and $|\,z-w|$.

(1 mark)

Show that for all z on the circle $|z|=1$

$$\left|\frac{\bar{z}\, e^{z} \cos z}{z^2+4}\right| \;\le\; \frac{e\cosh 1}{3}\,.$$

(8 marks)

(ii) Find all the solutions of the equation

$$e^{3z}+1+i=0.$$

(3 marks)

(iii) The path γ consists of the line segment from 1 to $1+i$, followed by the line segment from $1+i$ to $3+i$, followed by the line segment from $3+i$ to 3.

Evaluate

(a) $$\int_\gamma \mathrm{Re} z\, dz\,,$$

(b) $$\int_\gamma \bar{z}\, dz\,,$$

(c) $$\int_\gamma (2z^3\sinh z^2 \;+\; 2z\cosh z^2)\, dz.$$

(13 marks)

2 (i) Define what is meant by the following two statements:

(a) The function f is differentiable at the point z_0 ;

(b) A function f is analytic in a region D. *(2 marks)*

Let

$$g(z) = \frac{z}{(e^z + 1)^2} \,.$$

Decide where g is analytic giving reasons for your answer. *(5 marks)*

(ii) State, without proof, the Cauchy-Riemann equations for a differentiable function. *(1 mark)*

Let $h(z) = 4|z|^2$. Show that at every point of $\mathbb{C}\backslash\{0\}$, the function h is not differentiable .

Is h differentiable at the origin ? Give reasons for your answer. *(8 marks)*

(iii) In each of the following cases, determine whether there is a function k analytic on $\mathbb{C}$ with $\mathrm{Re}\,(k(x+iy)) = u(x,y)$, giving reasons for your answers:

$$\text{(c)} \quad u(x,y) = x^4 + \cosh x + 3\cosh y \,,$$

$$\text{(d)} \quad u(x,y) = x^3 - 3xy^2 + 2x + 1 \,.$$

When k exists, find an explicit expression for $k(z)$ in terms of z and show that you have found all the functions satisfying the conditions. *(9 marks)*

3 State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. ***(7 marks)***

Let γ be the circular contour $|z-2| = 4$ described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

(i) $\displaystyle\int_\gamma \frac{z\cosh z}{z-3}\,dz$, (ii) $\displaystyle\int_\gamma \frac{\sinh z}{z^2-16}\,dz$,

(iii) $\displaystyle\int_\gamma \frac{z\,e^z\,\cosh z}{z^2+25}\,dz$, (iv) $\displaystyle\int_\gamma \frac{\cos z}{(3z-1)^4}\,dz$,

(v) $\displaystyle\int_\gamma \mathrm{Re}\, z\,dz$.

(18 marks)

4 (i) State Liouville's Theorem. ***(2 marks)***

The function f is analytic in the complex plane and satisfies the relation

$$|f(z) + z| \; < \; |f(z) - z|$$

for all $z \in \mathbb{C}$.

Show that $f(z) \; = \; kz$, where k is a constant. ***(5 marks)***

(ii) For each of the following functions, find **all the singularities** in $\mathbb{C}$. Classify these singularities giving reasons for your answers and evaluate the residue at each of them:

(a) $\dfrac{1 - \cos \pi z}{z^3}$; ***(6 marks)***

(b) $\dfrac{z^2}{(e^z - 1)^2}$; ***(3 marks)***

(c) $(z - 1) \exp \left(\dfrac{1}{z + 1} \right)$; ***(5 marks)***

(d) $\dfrac{\cosh z + z^4}{z^7}$. ***(4 marks)***

5 (i) Let γ be the rectangular contour with vertices $-i$, i, $1+i$, $1-i$ described in the positive direction. Evaluate

$$\int_\gamma \frac{\sinh z}{\cos \pi z}\, dz\ , \qquad \int_\gamma \frac{z}{1+e^{2\pi i z}}\, dz\ ,$$

using Cauchy's Residue Theorem. ***(14 marks)***

(ii) Let $\alpha > 0$. Prove that

$$\int_{-\infty}^{\infty} \frac{(x+3)\cos \alpha x}{x^2-2x+5}\, dx \ = \ \frac{\pi}{e^{2\alpha}}\,(2\cos\alpha - \sin\alpha)\ .$$

(11 marks)

End of Question Paper