SCHOOL OF MATHEMATICS AND STATISTICS

Differential Geometry

Answer four questions. If you answer more than four questions, only your best four will be counted.

A list of formulae is provided on the last three pages.

Please leave this exam paper on your desk
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Registration number from U-Card (9 digits)
to be completed by student
Below is a list of true/false and multiple response questions. Each correct answer is worth 1 mark; you are required to justify your answers, and correct justifications are worth the remaining marks.

(i) True or false: The curve \( \gamma : \mathbb{R} \to \mathbb{R}^2, t \mapsto (t, |t|) \) is smooth. \(3 \text{ marks}\)

(ii) Which of the following properties can a map of the Earth have?
   (a) It can correctly represent all distances.
   (b) It can correctly represent all angles.
   (c) It can correctly represent all areas.
   \(6 \text{ marks}\)

(iii) Assume that a parametrised curve \( \gamma : \mathbb{R} \to \mathbb{R}^2 \) has a curvature function \( \kappa \). We reparametrise \( \gamma \) with the function \( \phi(t) = -t \) to get a new curve \( \tilde{\gamma} \). Which of the following is the curvature function \( \tilde{\kappa} \) of \( \tilde{\gamma} \)?
   (a) \(-\kappa(t)\)
   (b) \(\kappa(t)\)
   (c) \(-\kappa(-t)\)
   (d) It is not possible to know.
   \(3 \text{ marks}\)

(iv) True or False: A surface of revolution is always regular. \(3 \text{ marks}\)

(v) Which of the following properties does the standard parametrisation of the torus as a surface of revolution have?
   (a) Conformal
   (b) Area-preserving or equiareal
   (c) Local isometry
   \(6 \text{ marks}\)

(vi) Let \( S \) be a surface of constant Gaussian curvature, and let \( Q \) be a geodesic quadrilateral on \( S \) with all four inner angles equal to \( \frac{\pi}{2} \). Then the Gaussian curvature of \( S \) must be which of the following:
   (a) positive,
   (b) negative or
   (c) zero?
   \(4 \text{ marks}\)
2 (i) By approximating a curve $\gamma$ with straight lines, briefly justify the formula for the arc length of a parametrised curve. (4 marks)

(ii) Explain the general method for reparametrising a smooth, regular curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ to have unit speed. (4 marks)

(iii) Let $\gamma$ be the curve $t \mapsto (t, \cosh t)$.
   (a) Compute the arc length from 0 to $b$. (4 marks)
   (b) Give the unit speed reparametrisation of $\gamma$. (3 marks)
   (c) Compute the curvature using both the original parametrisation and your unit speed reparametrisation. (7 marks)
   (d) Does $\gamma$ have the same curvature as $\tilde{\gamma}(t) = (\cosh t, t)$? Justify your answer. (3 marks)

3 Let $U \subset \mathbb{R}^2$ be the open subset $\{(x, y) : 1 < \sqrt{x^2 + y^2} < 2\}$, let $V \subset \mathbb{R}^2$ be the open subset $\{(x, y) : 1 < x < 2\}$ and let $C \subset \mathbb{R}^3$ be the open cylinder $\{(x, y, z) : x^2 + y^2 = 1, 0 < z < 1\}$. Consider the following functions.
   $$\sigma_1 : V \rightarrow U \quad \sigma_1(x, y) = (x \cos y, x \sin y)$$
   $$\sigma_2 : U \rightarrow C \quad \sigma_2(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, \sqrt{x^2 + y^2} - 1\right)$$
   $$\sigma_3 : V \rightarrow C \quad \sigma_3(x, y) = (\cos y, \sin y, x - 1)$$

(i) Verify that $\sigma_3 = \sigma_2 \circ \sigma_1$. (2 marks)

(ii) Define what it means for $\rho : X \rightarrow \mathbb{R}^3$ to be a local isometry, and for what it means for $\rho : X \rightarrow \mathbb{R}^3$ to be a conformal map. Give necessary and sufficient conditions on the first fundamental form of $\rho$ for it to be a local isometry. Give necessary and sufficient conditions on the first fundamental form of $\rho$ for it to be a conformal map. (4 marks)

(iii) Compute the first fundamental form for each of the three $\sigma_i$ above. Are these maps local isometries? Are they conformal? (12 marks)

(iv) True or false: if $g \circ f$ is a local isometry, then both $g$ and $f$ are local isometries. Justify your answer. (3 marks)

(v) Find the surface area on $C$ of the image of the open box
   $$\left[\frac{5}{4}, \frac{7}{4}\right] \times \left[0, \frac{\pi}{4}\right]$$
   in $V$ under the map $\sigma_3$. Now find the surface area on $U$ of the image of the open box
   $$\left[\frac{5}{4}, \frac{7}{4}\right] \times \left[0, \frac{\pi}{4}\right]$$
   in $V$ under the map $\sigma_1$. (4 marks)
(i) Let $\sigma : U \to \mathbb{R}^3$ be a parametrisation of a surface. Define both the tangent space and the tangent plane at a point $(u_0, v_0) \in U$. Make sure the difference between the two concepts is clear. \hfill (5 marks)

(ii) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a smooth, regular curve in the $xz$-plane:

$$t \mapsto (x(t), 0, z(t)).$$

Define the surface of revolution generated by $\gamma$, and give its standard parametrisation using $t, \theta$. \hfill (4 marks)

(iii) Now let $\gamma_\epsilon(t) = (\epsilon \cos t, 0, \sin t)$ for $\epsilon > 1$. What is the surface of revolution $\Gamma_\epsilon$ generated by this curve, and how does it change as we vary $\epsilon$? \hfill (2 marks)

(iv) Identify the tangent space of the surface of revolution $\Gamma_\epsilon$ at a general point $(t, \theta)$. How does this change as we vary $\epsilon$? \hfill (4 marks)

(v) Compute the first fundamental form, the second fundamental form, and the Gaussian curvature of the surface of revolution $\Gamma_\epsilon$ at a general point $(t, \theta)$ under the standard parametrisation. How does the Gaussian curvature change at the points in the image of $(\pi, \theta)$ as we vary $\epsilon$? \hfill (6 marks)

(vi) Is the surface of revolution $\Gamma_\epsilon$ locally isometric to either the plane $\mathbb{R}^2$ or the sphere $S^2$? Justify your answer. \hfill (4 marks)
(i) Define the Euler characteristic $\chi(S)$ of a surface $S$. Be sure to explain what each of the terms in your formula is, any additional data needed to compute your formula and whether the answer is dependent or independent of those choices. (6 marks)

(b) Find a triangulation of a genus two surface (pictured below) and compute its Euler characteristic. (6 marks)

(ii) (a) State the classification of plane curves. (3 marks)

(b) Show that curves in $\mathbb{R}^2$ with zero curvature are precisely the straight lines. (3 marks)

(iii) (a) Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ be the surface parametrised by

\[(u, v) \mapsto (\alpha_1 u + \beta_1 v + \delta_1, \alpha_2 u + \beta_2 v + \delta_2, \alpha_3 u + \beta_3 v + \delta_3)\]

for coefficients $\alpha_i, \beta_i, \delta_i \in \mathbb{R}$. Compute the Weingarten matrix at a general point $(u, v)$, and show that the Gaussian curvature $K$ is zero. (4 marks)

(b) Are there surfaces in $\mathbb{R}^3$ with $K = 0$ which are not planes? Justify your answer. (3 marks)

End of Question Paper
List of formulae

• The inverse of a $2 \times 2$-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficients in $\mathbb{R}$ and $ad - bc \neq 0$ is
  \[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \]

• The cross-product of 2 vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ is
  \[ v_1 \times v_2 = (y_1z_2 - z_1y_2, z_1x_2 - z_2x_1, x_1y_2 - x_2y_1) \in \mathbb{R}^3. \]

• The angle $\theta$ between two vectors $v_1$ and $v_2 \in \mathbb{R}^3$ is given by
  \[ \cos \theta = \frac{v_1 \cdot v_2}{\|v_1\| \|v_2\|}. \]

Inverse hyperbolic functions are given by the following.

• $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$
• $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$
• $\tanh^{-1}(x) = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right)$ for $|x| < 1$

The derivatives of the inverse hyperbolic functions are given by the following.

• $\frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{x^2 + 1}}$
• $\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sqrt{x^2 - 1}}$ for $x > 1$
• $\frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1 - x^2}$ for $|x| < 1$

For a curve on $\mathbb{R}^2$ parametrised by $\gamma: [\alpha, \beta] \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$:

• The arc length from $\gamma(a)$ to $\gamma(b)$, $\alpha < a \leq b < \beta$ is:
  \[ \int_a^b \|\dot{\gamma}(t)\| \, dt \]

• The curvature of $\gamma$ at $\gamma(t)$ is
  \[ \kappa(t) = \frac{\ddot{\gamma}(t) \cdot J(\dot{\gamma}(t))}{\|\dot{\gamma}(t)\|^3} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}}, \]
  where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the counterclockwise rotation of angle $\pi/2$.

For a parametrised surface $\sigma: U \to \mathbb{R}^3$, with $U$ an open set in $\mathbb{R}^2$:
• The first fundamental form is given by

\[ I_{(u,v)} = \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix} \]

for all \((u,v) \in \mathbb{R}^2\), with \(E = \sigma_u \cdot \sigma_u\), \(F = \sigma_u \cdot \sigma_v\) and \(G = \sigma_v \cdot \sigma_v\).

• Area of the domain \(\sigma([\alpha_1, \beta_1] \times [\alpha_2, \beta_2])\), for \([\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subseteq U\):

\[ \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \sqrt{EG - F^2} \, dv \, du \]

• The preferred unit normal vector along \(\sigma\) is given by \(n: U \to \mathbb{R}^3\),

\[ n = \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||} \]

• The second fundamental form of \(\sigma\) at \((u,v) \in U\) is

\[ \Pi_{(u,v)} = \begin{pmatrix} L(u,v) & M(u,v) \\ M(u,v) & N(u,v) \end{pmatrix} \]

where \(L = \sigma_{uu} \cdot n\), \(M = \sigma_{uv} \cdot n\) and \(N = \sigma_{vv} \cdot n\).

• The Weingarten matrix of \(\sigma\) is

\[ W = I - \frac{1}{K} \Pi \]

• The Gaussian curvature is

\[ K = \det W. \]

The Gauss equations are

\[ EK = (\Gamma^1_{11})_u - (\Gamma^2_{12})_u + \Gamma^1_{11} \Gamma^2_{12} + \Gamma^1_{12} \Gamma^2_{22} - \Gamma^1_{12} \Gamma^2_{11} - (\Gamma^2_{12})^2 \]
\[ FK = (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12} \Gamma^1_{11} - \Gamma^2_{11} \Gamma^1_{22} \]
\[ FK = (\Gamma^2_{12})_v - (\Gamma^2_{11})_u + \Gamma^1_{22} \Gamma^2_{11} - \Gamma^2_{22} \Gamma^1_{11} \]
\[ GK = (\Gamma^2_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22} \Gamma^1_{11} + \Gamma^2_{22} \Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{12} \Gamma^1_{22}. \]

with

\[ \Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_u}{2(EG - F^2)}, \quad \Gamma^1_{12} = \frac{2EF_u - E^2_v - FE_u}{2(EG - F^2)}, \]
\[ \Gamma^2_{12} = \frac{GE_v - FG_u}{2(EG - F^2)}, \quad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}. \]

The Gauss-Bonnet formula for a geodesic triangle \(\Delta\) on a surface is

\[ \alpha_1 + \alpha_2 + \alpha_3 = \pi + \iint_\Delta K \, dA \]

where \(\alpha_1, \alpha_2, \alpha_3\) are the interior angles of \(\Delta\).

The Gauss-Bonnet formula for a compact surface \(S\) is

\[ \iint_S K \, dA = 2\pi \chi(S). \]