The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2016–17

Game Theory

2 hours and 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

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1 (i) There are $n \geq 1$ people working on a project. If the ith person contributes $x_i \geq 0$ hours $(1 \leq i \leq n)$, the resulting utility for each individual j is given by

$$u_j(x_1, \ldots, x_n) = \frac{100 \sum_{i=1}^{n} x_i}{1 + \sum_{i=1}^{n} x_i} - x_j$$

All individuals choose their efforts simultaneously aiming to maximize their utility.

(a) Find all pure-strategy Nash equilibria, and find the unique symmetric pure-strategy Nash equilibrium, i.e., an equilibrium in which all people contribute the same number of hours x. Find the value of x, and the utility for each player in that Nash equilibrium.

(8 marks)

(b) Suppose that the individuals can sign a contract agreeing on their (equal) contribution to the project. If the penalty for not honouring the agreement is so high that we are certain everyone will honour it, what should the agreed effort be, and what utility would each person obtain? ***(5 marks)***

(ii) Alice and Bob play a game given in strategic form as follows:

	I	II	III
A	2, 1	3, 4	5, 3
B	4, 3	1, 2	2, 1
C	3, 8	0, 6	1, 6

(a) Find all weakly dominated strategies and all strictly dominated strategies. ***(2 marks)***

(b) Eliminate iteratively all strictly dominated strategies. ***(2 marks)***

(c) Find all pure-strategy and all mixed-strategy Nash equilibria of this game. ***(8 marks)***

2 (i) Consider a finite, two-player, zero-sum game $G = (S, T, u)$.

(a) Describe the sets Δ^R and Δ^C of mixed strategies of both players. ***(2 marks)***

(b) Define the value of the game G. ***(2 marks)***

(c) Define what optimal strategies of G are. ***(2 marks)***

(d) Let p^* and q^* be optimal mixed strategies for the row and column players, respectively. Show that (p^*, q^*) is a Nash equilibrium. ***(7 marks)***

(ii) A magic square is an $n \times n$ array where each integer from 1 to n^2 occurs and with the property that all row and column sums are equal. For example,

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

is a 4×4 magic square.

Consider a two-player zero-sum game G whose payoff matrix for the row player is a $n \times n$ magic square. Find the value of G and find optimal strategies for both players. Explain your answer in detail. ***(12 marks)***

3 (i) Alice and Bob, the top game theorists in Sheffield, decide to provide game theory consultancy services. Alice first needs to decide whether to open an office in Sheffield or to move to Leeds; if she moves, both will earn 5 million pounds during the next twenty years. Bob must stay in Sheffield and if Alice also stays he will decide whether to cooperate with Alice or not. If Alice stays, she will observe Bob's decision, and then she would have to decide whether to cooperate or not. If Alice stays, their earning over the next twenty years (in millions of pounds) are 6 to both if they both cooperate, 2 to both if neither cooperates, 0 to Bob and 4 to Alice if Bob cooperates and Alice does not, 0 to Alice and 4 to Bob if Alice cooperates and Bob does not.

(a) Describe this game using a tree, carefully labelling all its components. ***(7 marks)***

(b) Solve this game using backward induction. ***(3 marks)***

(c) Describe the game in strategic form, find all its pure-strategy Nash equilibria and indicate which of these is subgame perfect. ***(7 marks)***

(ii) Consider a finite, two-player sequential game with corresponding rooted directed tree T.

(a) Define what is the rank of this game. ***(1 mark)***

(b) Prove Zermello's Theorem: for any subset S of the set of all outcomes of the game

(a) player I can force an outcome in S, or

(b) player II can force an outcome not in S.

(7 marks)

4 (i) Alice (the row-player) and Bob (the column-player) play repeatedly the game G given in tabular form as follows

	A	B
I	2, 2	0, x
II	x, 0	1, 1

where $x > 2$. After each repetition of the game, the game is played again with probability $1/2$.

(a) What are the expected payoffs for both players if the strategy profile (I, A) is played repeatedly. ***(2 marks)***

(b) If Alice always plays II, what is the largest expected payoff that Bob can get in this repeated game? ***(2 marks)***

(c) Consider the strategy profile for the repeated game in which Alice always plays I and Bob always plays A. Is this a Nash equilibrium? Justify your answer. ***(2 marks)***

(d) Consider the strategy profile for the repeated game in which Alice always plays II and Bob always plays B. Is this a Nash equilibrium? Justify your answer. ***(2 marks)***

(e) For which values of x can you find a Nash equilibrium for the repeated game with expected payoffs as in (a). Describe this Nash equilibrium in detail. ***(6 marks)***

(ii) An election is held to determine whether the Romulan state should remain in or exit the Galactic Union, and the outcome depends entirely on the votes of Alice and Bob, who can vote or abstain. The Romulan economy can be in one of two states, *Up* or *Down*; if the state is *Up*, the Romulans would be better off remaining and if it is *Down* they would be better off leaving. If the vote is a tie (including when both Alice and Bob abstain), both Alice and Bob get a payoff of $1/2$. If the vote is for remain and the economy is *Up* or if the vote is to leave and the economy is *Down*, both Alice and Bob get a payoff of 1. In all other cases the payoffs are 0 for both. Alice knows the state of the economy, but Bob does not and he believes that it is *Up* with probability $9/10$.

(a) Model this as a Bayesian game. ***(3 marks)***

(b) Show that the following two situations each describe a Bayes-Nash equilibrium.

- Alice abstains when the economy is *Up*, votes exit when the economy is *Down*, and Bob votes remain.
- Alice votes remain when the economy is *Up*, votes exit when the economy is *Down*, and Bob abstains.

(8 marks)

End of Question Paper