



The
University
Of
Sheffield.

MAS362

SCHOOL OF MATHEMATICS AND STATISTICS

**Autumn Semester
2016–17**

Financial Mathematics

2 hours and 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

**Please leave this exam paper on your desk
Do not remove it from the hall**

Registration number from U-Card (9 digits)
to be completed by student

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- 1 (i) Consider the following four bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.5	0	99.50
1	6%	104.71
1.5	10%	112.58
1.5	8%	?

- (a) Use the bootstrap method to find the 0.5, 1 and 1.5-year spot interest rates. **(9 marks)**
- (b) Find the price of the fourth bond in the table above. **(4 marks)**
- (ii) Consider a twelve-month forward contract on 1,000,000 shares of *Too Big To Fail Plc*. These shares are currently traded for £80 per share. Within the next twelve months, *Too Big To Fail Plc* will pay a single dividend of £3 per share in 6 months. The six-month and twelve-month spot interest rates are 2% and 3%, respectively.
- (a) What is the correct forward price for this forward contract? **(2 marks)**
- (b) You are given the opportunity to take a long or a short position in this forward contract at a forward price of £79,000,000. Describe in detail an arbitrage opportunity available to you. **(10 marks)**

- 2 (i) Let $0 \leq X_1 < X_2$ and let $0 \leq \lambda \leq 1$. Consider three European call options on the same underlying asset and with the same expiration time T . The strike prices of these are X_1 , X_2 and $\lambda X_1 + (1 - \lambda)X_2$ and their corresponding spot prices are denoted c_1 , c_2 and c .

For any $0 \leq \lambda \leq 1$ consider portfolio Π_λ which is

- long λ call options with strike X_1 ,
- long $1 - \lambda$ call options with strike X_2 ,
- short one call option with strike $\lambda X_1 + (1 - \lambda)X_2$.

- (a) Sketch the graph of the payoff of $\Pi_{\frac{1}{2}}$ as a function of the underlying asset price S_T at expiration. **(4 marks)**

- (b) Show that for any S ,

$$\max\{S - \lambda X_1 - (1 - \lambda)X_2, 0\} \leq \lambda \max\{S - X_1, 0\} + (1 - \lambda) \max\{S - X_2, 0\}.$$

You may want to show that this holds in the following three cases separately: $S < X_1$, $X_1 \leq S \leq X_2$, $X_2 < S$ and use the fact that $X_1 \leq \lambda X_1 + (1 - \lambda)X_2 \leq X_2$. **(6 marks)**

- (c) Write down the payoff function of Π_λ at expiration and use part (b) to deduce an inequality satisfied by the payoff. **(3 marks)**

- (d) Use part (c) to deduce an inequality between c_1 , c_2 and c . **(6 marks)**

- (ii) The price of a stock which pays no dividends is currently £8. Over each of the next two 1-year periods the stock price will either increase by 50% or decrease by 50%. Suppose that all interest rates are constant and equal to 2%.

Use a binomial tree to find the price of a two-year American put option on this stock with strike price £6. **(6 marks)**

- 3 (i) (a) State Ito's Lemma. *(3 marks)*
- (b) Let $\{B_t\}_{t \geq 0}$ denote Brownian motion. Describe the stochastic process $\{B_t^2\}_{t \geq 0}$. *(4 marks)*

- (ii) Consider two assets whose prices S_1 and S_2 follow the Ito processes

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dB, \quad dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dB$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 \geq 0$ and B denotes Brownian motion. Assume that all risk-free interest rates are constant and equal to r .

- (a) Describe a portfolio Π involving these assets whose value after an infinitesimal period of time is known in advance. Explain your answer. *(6 marks)*
- (b) At time t you invest in one unit of Π ; what is the approximate value of your investment after a very short period of time Δt . *(2 marks)*
- (c) At time t you invest the value of Π in a risk-free investment; what is the approximate value of your investment after a very short period of time Δt . *(4 marks)*
- (d) Use (b) and (c) to describe an identity involving $\mu_1, \mu_2, \sigma_1, \sigma_2$ and r . *(6 marks)*
- 4 (i) Explain the following terms in the context of portfolio theory:
- (a) *feasible set*, *(2 marks)*
- (b) *efficient frontier*, and *(2 marks)*
- (c) *market portfolio*. *(2 marks)*
- (ii) Consider a market with only two risky investments A and B. Let their expected returns be 12% and 20%, respectively. Their standard deviation of returns are 10% and 20%, respectively. The correlation between the returns of A and B is 0.75.
- (a) Assuming there is no risk-free investment, find the feasible set as a curve in the σ - r plane given in parametric form. What is the efficient frontier? *(7 marks)*
- (b) Assume that a risk-free investment in this world exists and has risk-free return equal to 3%. Find the market portfolio. *(12 marks)*

End of Question Paper