



SCHOOL OF MATHEMATICS AND STATISTICS

**Autumn Semester
2016–17**

MAS377 Mathematical Biology

2 hours

*Marks will be awarded for your best **three** answers.*

- 1 It is proposed that the dynamics of lions (L) and zebras (Z) in a national park can be given by the following ordinary differential equations,

$$\frac{dZ}{dt} = (a - Z)Z - bLZ \quad (1)$$

$$\frac{dL}{dt} = \theta bLZ - cL, \quad (2)$$

where a, b, c, θ are positive constants.

- (i) Find the two (non-trivial) equilibria of this system, showing that the equilibrium zebra population is either $Z_1 = a$ or $Z_2 = c/(\theta b)$. **(5 marks)**
- (ii) Calculate the Jacobian, J , of the system at the general equilibrium (Z_*, L_*) . Substitute in each of the two equilibria from part (i), and show how the stability of each equilibrium depends on b . **(12 marks)**
- (iii) Show that whenever the lion population stably co-exists with the zebras, zebra numbers are lower than if the lions were not present. **(3 marks)**
- (iv) The rate of predation *per lion* is bZ .
- (a) Explain why this might not be realistic for large populations of zebras, and suggest an alternative form for the predation rate. **(3 marks)**
- (b) Suggest a biological interpretation for the parameter θ . **(2 marks)**

- 2 A population of insects is exposed to an infectious disease. Partitioning the population in to either susceptible (S) or infected (I) compartments, the dynamics of this population are given by the ordinary differential equations,

$$\begin{aligned}\frac{dS}{dt} &= bS - \beta SI - dS \\ \frac{dI}{dt} &= \beta SI - \alpha(S + I)I - dI,\end{aligned}$$

where b is the birth rate, d the death rate, β the transmission coefficient and α the virulence (death due to infection) coefficient, and we assume $b > d$.

- (i) What is the *per capita* rate of virulence experienced by an infected insect? What does this mean biologically? **(2 marks)**
- (ii) (a) Calculate expressions for the nullclines of the system, i.e. where either $dS/dt = 0$ or $dI/dt = 0$. Sketch these nullclines, along with qualitative directions of flow and example trajectories to form two phase portraits for this system, one for the case $\beta - \alpha < 0$ and one for the case $\beta - \alpha > 0$. **(10 marks)**
- (b) Inferring from your phase portraits, describe the long-term outcomes of the two cases. **(2 marks)**
- (iii) What is meant by the term ‘endemic equilibrium’? **(2 marks)**
- (iv) Calculate the Jacobian for this system and thus confirm that the endemic equilibrium is both feasible and stable if and only if $\beta - \alpha > 0$. (*Hint: it is not necessary to calculate the equilibrium values for S and I , though you may if you wish.*) **(9 marks)**

- 3** A model for the regulation of expression of a gene is given by

$$\frac{dM}{dt} = -\mu M + f(t),$$

where $M(t)$ represents the amount of mRNA in a cell, μ is a positive constant, and $f(t) \geq 0$.

- (i) What do μ and $f(t)$ represent? **(2 marks)**
- (ii) If $M(0) = M_0 > 0$ and

$$f(t) = \begin{cases} K, & 0 \leq t < T \\ 0, & t \geq T \end{cases}$$

where $K > 0$ is a positive constant, find an expression for $M(t)$ for $t \geq 0$. Show that $M(2T) = M(T)e^{-\mu T}$. Sketch $M(t)$ when $M_0 < K/\mu$.

(10 marks)

- (iii) Show that if $M(2T) = M_0$ then $M_0 = M_* \equiv \frac{K(1 - e^{-\mu T})}{2\mu \sinh \mu T}$.

Sketch $M(t)$ when M_0 takes this value, showing where $M(2T) = M_0$.

(5 marks)

- (iv) Consider a ‘square-wave’ form for $f(t)$, where

$$f(t) = \begin{cases} K, & 2nT \leq t < (2n + 1)T \\ 0, & (2n + 1)T \leq t < (2n + 2)T \end{cases}$$

for $n = 0, 1, 2, 3, \dots$

If $M_0 = M_*$, use the results of parts (ii) and (iii) to explain why $M(t)$ is periodic for this form of $f(t)$. What is the period of $M(t)$? Sketch $f(t)$ and $M(t)$.

Show that the fold-change in the mRNA expression level (the ratio of the maximum and minimum level of $M(t)$) is $\rho = e^{\mu T}$.

Calculate ρ if $\mu = 0.03\text{min}^{-1}$ and $T = 60\text{min}$.

(8 marks)

- 4 (i) Explain what is meant by bistability. Why is bistability in gene expression important for cells? *(2 marks)*
- (ii) A model for the interaction of two genes is given by

$$\frac{dX}{dt} = f(Y) - \mu X \quad (3)$$

$$\frac{dY}{dt} = f(X) - \mu Y, \quad (4)$$

where X and Y represent the amounts of the products of the two genes, $f(X) = \frac{\theta^2}{\theta^2 + X^2}$, and μ and θ are positive constants.

- (a) Sketch the nullclines of the system when (A) $\theta < 1/(2\mu)$, and (B) $\theta > 1/\mu$. State how many steady states the system has in each case. *(5 marks)*
- (b) Let (X_*, Y_*) be a steady state of the system. By linearising, show that (X_*, Y_*) is a stable node if $f'(X_*)f'(Y_*) < \mu^2$ and a saddle point if $f'(X_*)f'(Y_*) > \mu^2$. *(5 marks)*
- (c) By finding expressions for the gradients of the nullclines at a steady state, show that the system is bistable when $\theta \ll 1/(2\mu)$. *(6 marks)*
- (d) Show that the system has a steady state (X_*, X_*) , where

$$\mu X_* = f(X_*),$$

and that at this steady state $\phi_X \phi_Y = 1$, where ϕ_X and ϕ_Y are the gradients of the X and Y nullclines, respectively. Hence show that when $\theta < 1/(2\mu)$ the system is bistable. *(7 marks)*

End of Question Paper