Data provided: Formula sheet

MAS380

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2016–17

Computational Engineering Mathematics — Three hours

Marks will be awarded for your best FOUR answers.
The maximum possible mark for the paper is 100.

1 (a) Second order linear partial differential equations for $u(x, y)$ can be written in the general form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

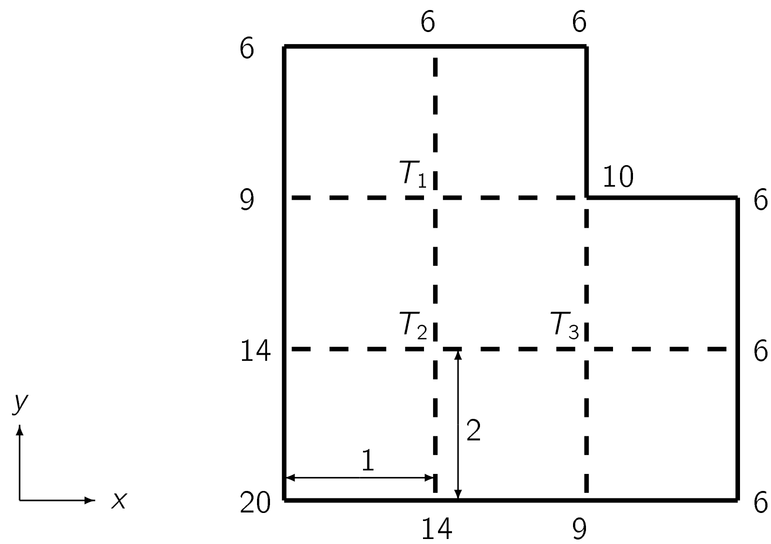
where A, B, C, D, E, F and G are functions of x and y.

(i) What are the conditions on A, B, C, D, E, F and G which determine whether the equation is elliptic, parabolic or hyperbolic? ***(3 marks)***

(ii) Identify the regions of the (x, y) plane where the equation

$$x^3\frac{\partial^2 u}{\partial x^2} + (y-1)\frac{\partial^2 u}{\partial y^2} = 0$$

is elliptic, parabolic or hyperbolic. ***(6 marks)***



1 (continued)

(b) The figure (where the lower left corner is the origin $x = y = 0$) shows the solution domain, divided into intervals of length $\Delta x = 1$ in the x direction, and length $\Delta y = 2$ in the y direction. $T(x, y)$ satisfies the indicated boundary conditions, and the differential equation

$$\frac{\partial^2 T}{\partial x^2} + x^2 \frac{\partial^2 T}{\partial y^2} = 0.$$

(i) Is this differential equation elliptic, parabolic or hyperbolic? ***(2 marks)***

(ii) Use the finite difference formulae on the formula sheet to find the finite difference equations required to find estimates of the nodal values T_1, T_2 and T_3. ***(7 marks)***

(iii) Express the finite difference equations in part (ii) in the form $A\mathbf{T} = \mathbf{b}$, where A is a 3×3 matrix, $\mathbf{T} = [T_1, T_2, T_3]^T$ and $\mathbf{b} = [82, 70, 25]^T$, where you should give the matrix A.

Hence, using Gaussian elimination or otherwise, show that

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \frac{1}{356} \begin{bmatrix} 3332 \\ 4128 \\ 3257 \end{bmatrix}.$$ ***(7 marks)***

2 (a) The temperature $T(x,t)$ satisfies the heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad (0 \leqslant x \leqslant 1, \quad t > 0),$$

and the boundary and initial conditions

$$T(0,t) = 0, \qquad \frac{\partial}{\partial x} T(x,t)\bigg|_{x=1} = 1, \qquad T(x,0) = x,$$

where all units are in terms of °C, m and s.

If $T_{i,j} = T(x_i, t_j)$, with $i = 0$ and $i = 5$ corresponding to $x = 0$ and $x = 1$, respectively, and $j = 0$ corresponding to $t = 0$, use backward differences for time derivatives and central differences for space derivatives to derive the equations for $T_{1,1}$, $T_{2,1}$, $T_{3,1}$, $T_{4,1}$ and $T_{5,1}$. Take $\Delta t = 0.04$. (You do *not* need to attempt to solve the equations.) ***(13 marks)***

(b) (i) Suppose that a tensor σ_{ij} satisfies

$$\varepsilon_{ijk}\sigma_{jk} = 0$$

for $i = 1, 2, 3$, where ε_{ijk} is the Levi-Civita tensor.

By considering all possible values of i, j and k show that

$$\sigma_{jk} = \sigma_{kj}$$

for all $j = 1, 2, 3$ and $k = 1, 2, 3$. ***(7 marks)***

(ii) Using index notation, show that for any vectors $\mathbf{u}$ and $\mathbf{v}$

$$\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}(\nabla\cdot\mathbf{v})+(\mathbf{v}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{v}-\mathbf{v}(\nabla\cdot\mathbf{u}).$$ ***(5 marks)***

(You may use $\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, where ε_{ijk} is the Levi-Civita tensor, and δ_{ij} is the Kronecker delta tensor.)

3 (a) A rectangular surface on a positive x-plane is defined by $x = 0$, $0 \leqslant y \leqslant 2$ and $0 \leqslant z \leqslant 1$ (where all lengths are measured in m). The stress tensor depends on z and is given by (in units of Pa)

$$[\sigma] = \begin{bmatrix} 2z & 3z^2 & 8z^3 \\ 3z^2 & 6z^2 & 4z \\ 8z^3 & 4z & 2z \end{bmatrix}.$$

Show that the total stress force $\mathbf{f}$ on the rectangular surface is

$$\mathbf{f} = 2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{N}.$$

(6 marks)

(b) Let the i-component of the body force per unit volume in a solid body be F_i, and the components of the stress tensor be σ_{ji}. Consider a volume V in the solid body which is enclosed by a surface S.

If the body is in equilibrium, derive an equation (involving integrals) for the total force on the volume V.

Using Gauss' theorem in the form

$$\int_V \frac{\partial \sigma_{ji}}{\partial x_j}\, dV = \oint_S \sigma_{ji} \hat{n}_j \, dS,$$

where dS is a surface area element and $\hat{\mathbf{n}}$ is a unit vector normal to S and pointing out of V, deduce that

$$\frac{\partial \sigma_{ji}}{\partial x_j} + F_i = 0$$

at every point in the body. ***(7 marks)***

(c) The stress tensor is given by

$$\sigma_{ij} = \lambda\, \delta_{ij}\, \varepsilon_{mm} + 2\mu\, \varepsilon_{ij},$$

where λ and μ are constants, δ_{ij} is the Kronecker delta tensor, and

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where u_i is the displacement in the ith coordinate direction.

(i) Show that the result of part (b) gives

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}+\mathbf{F} = \mathbf{0}.$$

(7 marks)

(ii) If $\mathbf{u} = (x_3^2, 0, 0)^T$, use the result of part (c)(i) to find the body force $\mathbf{F}$. ***(5 marks)***

4 (a) A quantity $f(\mathbf{x}, t)$ is defined in a fluid at time t, at a point with position vector $\mathbf{x} = (x, y, z)^T$.

By considering the change in f, moving with the fluid, between times t_1 and t_2 (where $t_2 - t_1$ is small), show that the substantial derivative (i.e. the rate of change with time, moving with the fluid) is given by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f,$$

where the fluid velocity is $\mathbf{v} = (u, v, w)^T$. ***(6 marks)***

(b) Consider an incompressible fluid where the density ρ is a constant.

Show from the continuity equation

$$\frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0$$

that

$$\nabla \cdot \mathbf{v} = 0,$$

where $\mathbf{v}$ is the fluid velocity.

Given that the stress tensor can be written as

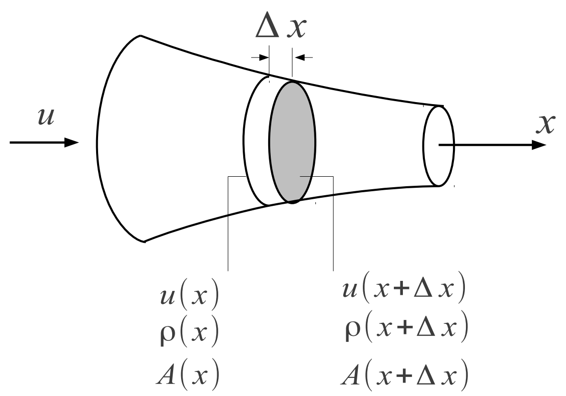
$$\sigma_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),$$

where μ is a constant, show that the momentum equation can be written as

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{F},$$

where p is the pressure and $\mathbf{F}$ is the body force per unit volume.

(10 marks)



(c) Consider the fluid motion in a pipe with varying cross-sectional area, as shown in the figure. The velocity and the density of the fluid do not vary in each cross-section. The x-axis points in the direction of the pipe, which is also the direction of the velocity. Let $A(x)$ be the area of the cross-section at x, $\rho(x)$ be the density, and $u(x)$ be the velocity.

 Question 4 continued on next page

4 (continued)

By considering the mass of the fluid in a fixed volume between the cross-sections at x and $x + \Delta x$, where Δx is small, derive the continuity equation in the following form:

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho u A) = 0.$$ ***(9 marks)***

[Hint: the mass of fluid passing through the cross-section at x in a unit time is $\rho(x)u(x)A(x)$.]

5 (a) The boundary value problem

$$\mathcal{L}(u) = x\frac{d^2u}{dx^2} + \frac{du}{dx} + u = 0 \qquad \text{for} \quad 0 \leqslant x \leqslant 1,$$

is to be solved by the weighted residual method, given that $u'(0) = 0$ and $u(1) = 1$ (where $u'(x) = du/dx$).

(i) Determine which of $U_1(x)$ and $U_2(x)$ can be taken as a trial function, where

$$U_1(x) = x + c_1x^2(1-x) + c_2x^3(1-x)$$

$$U_2(x) = 1 + c_1x^2(1-x) + c_2x^2(1-x^2),$$

c_1 and c_2 being constants. ***(6 marks)***

(ii) Determine the residual $R(x) = \mathcal{L}(U)$ associated with your chosen trial function. ***(4 marks)***

Use the condition

$$\int_0^1 w(x)R(x)\,dx = 0,$$

with the weight functions $w(x) = 1$ and $w(x) = x$, to derive two equations for c_1 and c_2. ***(6 marks)***

Show that the solution to these equations is

$$c_1 = \frac{3060}{61}, \qquad c_2 = -\frac{1470}{61}.$$ ***(4 marks)***

(b) Given the equation

$$\frac{d}{dx}\left\{A(x)\frac{du}{dx}\right\} + f(x) = 0 \qquad \text{for} \quad 0 \leqslant x \leqslant \ell,$$

where $A(x)$ and $f(x)$ are known functions, and $u(x)$ is to be determined, show that the weak form of the equation is

$$\int_0^\ell A\,\frac{du}{dx}\,\frac{dw}{dx}\,dx = w(\ell)F(\ell) - w(0)F(0) + \int_0^\ell w(x)f(x)\,dx,$$

where $F(x) = A(x)\dfrac{d}{dx}u(x)$, and $w(x)$ is the weight function. ***(5 marks)***

End of Question Paper

Formula Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,j+1} - U_{i,j}}{\Delta t}$$

Forward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,j} - U_{i,j}}{\Delta x}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,j} - U_{i,j-1}}{\Delta t}$$

Backward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i,j} - U_{i-1,j}}{\Delta x}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2}(x_i,\ t_j) \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2}$$

Relation between different parameters:

A number of relationships between E, ν, K, λ and μ hold and are summarized in Table 1. μ ($\equiv G$) is the elastic shear modulus, K the elastic bulk modulus, E the elastic stiffness (or Young's Modulus) and ν Poisson's ratio.

	E	ν	K	λ	$\mu \equiv G$
E, ν	-	-	$\frac{E}{3(1-2\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$
E, K	-	$\frac{3K-E}{6K}$	-	$\frac{K(9K-3E)}{9K-E}$	$\frac{3KE}{9K-E}$
K, μ	$\frac{9\mu K}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	-	$K - \frac{2\mu}{3}$	-

Table 1: The relations between the properties of elastic bodies.