Data provided: Formula sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016–17**

Mathematics III (Electrical) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 Find the image in the w-plane of the following point, lines and circle when the mapping rule $w = 1/(z-1)$ is applied to:

(i) The point $z = 1 + j$. ***(3 marks)***

(ii) The line $z = re^{j\pi/3} + 1$. ***(5 marks)***

(iii) The line $z = 2 + re^{-j\pi/2}$. ***(13 marks)***

(iv) The circle $z = e^{j\theta} + 1$. ***(4 marks)***

Sketch your results in the z and w-planes. For questions (ii) and (iii) $0 \le r < \infty$, while for (iv) $0 \le \theta < 2\pi$.

2 (i) Represent graphically the set of values of z for which

$$\left| \frac{z-3}{z+3} \right| > 2.$$

(7 marks)

(ii) Expand the function

$$f(z) = \frac{1-z}{z^2 - \left(a + \frac{1}{a}\right) z + 1}, \quad a > 1,$$

into a Laurent series in the interval $\dfrac{1}{a} < |z| < a$. ***(18 marks)***

3 (i) Consider the vector field $\mathbf{A} = (2xy^3 - z^2, 3x^2y^2 + z, y - 2xz)$.

(a) Calculate div $\mathbf{A}$ and show that curl $\mathbf{A} = 0$. ***(4 marks)***

(b) By evaluating both sides, verify that

$$\nabla^2 \mathbf{A} = \text{grad div}\mathbf{A} - \text{curl curl}\mathbf{A}.$$

(6 marks)

(c) Find a scalar field, $\Phi = \Phi(x, y, z)$, such that

$$\mathbf{A} = \text{grad}(\Phi).$$

(7 marks)

(ii) Calculate

$$\oint_{\mathcal{C}} \frac{z^2 - 2z}{(z+1)^2(z^2+4)} dz$$

if $\mathcal{C}$ is the circle $|z - 2j| = 5$. ***(8 marks)***

4 (i) Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (x^2 + 2y)\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ and $\mathcal{C}$ is the curve from $(0, 0, 0)$ to $(4, 4, 8)$ given in parametric form by $\mathcal{C} : 2t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. Choose appropriate values for t of the given endpoints of $\mathcal{C}$. ***(10 marks)***

(ii) Let S be the surface given by $z = x^2 + y^2$, with $0 \leq z \leq 4$, and let C be the boundary of S. Consider the vector field $\mathbf{F} = x\mathbf{i} + 2x\mathbf{j} + 3x\mathbf{k}$. Prove Stokes' theorem when the boundary C is traversed in the counterclockwise direction. ***(15 marks)***

End of Question Paper

Formula sheet

- The general formula for the residue at a pole z_0, of order m is

$$\frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right\}$$

- Useful identities

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}; \quad \sin m\theta \cos n\theta = \frac{1}{2}[\sin(m+n)\theta + \sin(m-n)\theta]$$

- The polar and spherical area elements are given by

$$dA = r dr d\theta, \quad dA = r^2 \sin\phi \, d\phi \, d\theta$$