The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016–17**

Topics in Advanced Fluid Mechanics **2 hours 30 minutes**

Marks will be awarded for your best ***four*** *answers.*

1 Consider the impulse equations for an incompressible fluid in the geometric gauge

$$\frac{D\boldsymbol{\gamma}}{Dt} = -(\nabla \boldsymbol{u})^T \boldsymbol{\gamma}.$$

By following the steps below, derive the Cauchy invariant

$$\gamma_i(\boldsymbol{a}, t) = \gamma_j(\boldsymbol{a}, 0)\frac{\partial a_j}{\partial x_i}, \quad i = 1, 2, 3$$

where the summation convention is assumed on repeated indices.

(i) Derive the equations for the Jacobian matrix

$$\frac{D}{Dt}\frac{\partial x_i}{\partial a_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial a_j}, \quad (i, j = 1, 2, 3.)$$

(5 marks)

(ii) Derive the equations for the inverse Jacobian matrix

$$\frac{D}{Dt}\frac{\partial a_i}{\partial x_j} = -\frac{\partial a_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}, \quad (i, j = 1, 2, 3.)$$

(5 marks)

(iii) Assume

$$\gamma_i(\boldsymbol{a}, t) = C_j(\boldsymbol{a}, t)\frac{\partial a_j}{\partial x_i}, \quad (i = 1, 2, 3)$$

and show that $\boldsymbol{C}$ is constant. ***(10 marks)***

(iv) State the Cauchy invariant for the equation for the vorticity $\boldsymbol{\omega}$ and show that $\boldsymbol{\gamma} \cdot \boldsymbol{\omega}$ is an invariant of motion. ***(5 marks)***

2 Consider the Burgers equation subject to a forcing term

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} + \frac{\partial N}{\partial x} = 0, \tag{1}$$

where $N(x,t)$ is a given function of x and t.

(i) By assuming $u = \dfrac{\partial \phi}{\partial x}$, show that the left-hand side can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} + \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - \nu\frac{\partial^2 \phi}{\partial x^2} + N\right].$$

(5 marks)

(ii) By taking $\phi = -2\nu \log \psi$, show that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} + \frac{\partial N}{\partial x} = -2\nu\frac{\partial}{\partial x}\left[\frac{\frac{\partial \psi}{\partial t} - \nu\frac{\partial^2 \phi}{\partial x^2} - \frac{N}{2\nu}\psi}{\psi}\right].$$

(10 marks)

(iii) Show that the equation (1) can be reduced to the following

$$\frac{\partial \psi}{\partial t} = \nu\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2\nu}N\psi. \tag{2}$$

(5 marks)

(iv) For a choice of the forcing $N(x) = 2\nu^2\left(1 - 2\tanh^2(x)\right)$ which is localised in space, show that $\psi(x) = \mathrm{sech}(x) \equiv (\cosh(x))^{-1}$ is a steady solution of (2). ***(5 marks)***

3 Consider the model equation for vorticity $\omega(x,t)$ defined on $x \in \mathbb{R}^1$:

$$\frac{\partial \omega}{\partial t} = \omega H[\omega],$$

with an initial condition $\omega(x,0) = \omega_0(x)$. Here $H[\omega]$ denotes the Hilbert transform $H[\omega](x) = \dfrac{1}{\pi} \displaystyle\fint_{-\infty}^{\infty} \frac{\omega(y)}{x-y} dy$, defined with a principal-value integral. We solve it using a Fourier series $\omega(x,t) = \displaystyle\sum_{n=-\infty}^{\infty} A_n(t) e^{inx}$, where $A_0(t) = 0$ for $\forall t \geq 0$ and $A_n(t) = A^*_{-n}(t)$ for $n \geq 1$, with $*$ indicating complex conjugation.

(i) Consider a splitting of ω into upper- and lower-analytic components $\omega(x,t) = \omega_+(x,t) + \omega_-(x,t)$, where
$\omega_+(x,t) = \displaystyle\sum_{n=1}^{\infty} A_n(t) e^{inx}$ and $\omega_-(x,t) = \displaystyle\sum_{n=1}^{\infty} A_{-n}(t) e^{-inx}$.
Show that

$$H[\omega] = -i(\omega_+ - \omega_-).$$

Hint: the Fourier transform of the Hilbert transform is given by $\widetilde{H}(k) = -i\,\mathrm{sgn}(k)$, where $\mathrm{sgn}(k) = \pm 1$ for wavenumber $k \gtrless 0$.

(6 marks)

(ii) Show that

$$\frac{\partial \omega_+}{\partial t} = -i\omega_+^2$$

and

$$\frac{\partial \omega_-}{\partial t} = i\omega_-^2$$

hold for each component. ***(6 marks)***

(iii) Solve the above equations for $\omega_+(x,t)$ and $\omega_-(x,t)$. ***(6 marks)***

(iv) Derive the solution $\omega(x,t)$ in the following form

$$\omega(x,t) = \frac{4\omega_0}{(2 - tH[\omega_0])^2 + (\omega_0 t)^2}.$$

(7 marks)

4 Consider motion of an elliptical vortex with uniform vorticity in two dimensions, which is defined by

$$f(x,y,t) = \frac{(x\cos\theta(t) + y\sin\theta(t))^2}{a(t)^2} + \frac{(-x\sin\theta(t) + y\cos\theta(t))^2}{b(t)^2} - 1,$$

subject to the time-independent velocity field $u(x,y) = ex$, $v(x,y) = -ey$. Here $a(t), b(t)$ denote the major and minor axes of the ellipse, $\theta(t)$ the angle that the major axis makes with the x-axis and e a constant strain-rate.

(i) Compute $\dfrac{\partial f}{\partial t}$, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. ***(6 marks)***

(ii) Write down the coefficients of x^2, y^2 and xy terms in the kinematic boundary conditions

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0.$$

(9 marks)

(iii) Derive the governing equations for $a(t), b(t)$ and $\theta(t)$ as follows:

$$\frac{da}{dt} = ea\cos 2\theta,$$

$$\frac{db}{dt} = -eb\cos 2\theta,$$

$$\frac{d\theta}{dt} = -e\frac{a^2+b^2}{a^2-b^2}\sin 2\theta.$$

(10 marks)

5 Consider motion of fluid particles subject to a two-dimensional incompressible Euler flow

$$\frac{d\boldsymbol{X}(t)}{dt} = \boldsymbol{u}(\boldsymbol{X}(t), t).$$

Let us denote the pair-separation by $\rho(t) = |\boldsymbol{X}(t) - \boldsymbol{Y}(t)|$, where $\boldsymbol{X}(t), \boldsymbol{Y}(t)$ denote particle paths starting from $\boldsymbol{X}(0), \boldsymbol{Y}(0)$ at $t = 0$, respectively.

(i) Derive the following inequality

$$\left|\frac{d\rho(t)}{dt}\right| \leq |\boldsymbol{u}(\boldsymbol{X}(t), t) - \boldsymbol{u}(\boldsymbol{Y}(t), t)|.$$

(5 marks)

(ii) Derive the following estimate

$$\left|\frac{d\rho(t)}{dt}\right| \leq C \sup_{\boldsymbol{x}} |\omega(\boldsymbol{x}, 0)| \rho(t) \ln \frac{eL}{\rho(t)}.$$

Hint: You may make use of the inequality

$$|\boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{u}(\boldsymbol{y}, t)| \leq C \sup_{\boldsymbol{x}} |\omega(\boldsymbol{x}, t)| |\boldsymbol{x} - \boldsymbol{y}| \ln \frac{eL}{|\boldsymbol{x} - \boldsymbol{y}|},$$

where C denotes a positive constant, L a characteristic length and $e = \exp(1)$.

(5 marks)

(iii) Derive the following upper- and lower- bounds for the pair-separation

$$\left(\frac{\rho_0}{eL}\right)^{\exp(Ct \sup_{\boldsymbol{x}} |\omega(\boldsymbol{x},0)|)} \leq \frac{\rho(t)}{eL} \leq \left(\frac{\rho_0}{eL}\right)^{\exp(-Ct \sup_{\boldsymbol{x}} |\omega(\boldsymbol{x},0)|)},$$

where $\rho_0 = \rho(0)$. ***(10 marks)***

(iv) Can two fluid particles collide in finite time or not? State (without proof) the implications of your answer on the mathematical property of two-dimensional Euler equations. ***(5 marks)***

End of Question Paper