Data provided: Formula Sheet

MAS420

The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2016–17**

**MAS420 Signal Processing** **2 hours**

*Attempt **ALL** questions.*

**Please leave this exam paper on your desk**
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**1** (i) In the Hilbert space of finite power signals of period $T$, under the inner product

$$(f, g) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) g^*(t)\, dt,$$

prove that the set $\phi_n(t) = e^{in\sigma t}; -\infty < t < \infty$, where $\sigma = 2\pi/T$, form an orthogonal set (both the orthogonality and unit norm conditions must be proved). ***(6 marks)***

(ii) Sketch the periodic function, f(t), of period 2, for $-3 \leq t < 3$, where

$$f(t) = \begin{cases} 1 & -1 < t \leq 0 \\ -1 & 0 < t \leq 1 \end{cases}$$

Find the complex Fourier coefficients for this signal. ***(8 marks)***

(iii) Use Parseval's theorem and your result in (ii) to derive the equality

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

***(5 marks)***

(iv) The signal is passed over a link which will not pass frequencies greater than $2\pi$ rad s$^{-1}$ but passes other frequencies unchanged The received signal is $g(t)$. Find an expression for g(t) as a sine/cosine series and show that during transmission approximately 19% of the power has been lost. ***(6 marks)***

**2** (i) Making use of clear sketches find $g(t) = p_a(t) * p_b(t)$, where $0 < a < b$, and $p_c(t)$ is the rectangular pulse of width $2c$, and calculate its energy $E(g)$.

***(9 marks)***

(ii) Prove the convolution theorem: if $f(t)$ and $g(t)$ are signals with Fourier Transforms $F(\omega)$ and $G(\omega)$ respectively, then

$$(f * g)(t) \longleftrightarrow F(\omega)G(\omega).$$

Use the theorem to find the Fourier Transform of the signal $g(t)$ of part (i).

***(7 marks)***

(iii) Prove the Plancherel theorem: if $f(t)$ and $g(t)$ are signals with Fourier transforms $F(\omega)$ and $G(\omega)$ respectively, then

$$(f, g) = \frac{1}{2\pi}(F, G).$$

Deduce that

$$\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = 2\pi E(f),$$

where $E(f)$ is the energy in the signal $f(t)$.

Use this and the results of part (i) (with a suitable choice of $a$ and $b$) to show that

$$\int_{-\infty}^{\infty} \text{sinc}^4(\omega) \, d\omega = \frac{2\pi}{3}.$$

***(9 marks)***

**3** (i) A signal $x(t)$ with Fourier Transform $X(\omega)$ is input to a system with system transfer function (STF) $H(\omega)$. Define the system transfer function. Write down the expression for the Fourier Transform, $Y(\omega)$, of the signal, $y(t)$, that is output from the system.

Given that $h(t)$, the impulse response function, is the inverse Fourier transform of the STF and the step response function, $s(t)$, is the response of a system to the unit step signal, $U(t)$, prove that $s(t) = \int_{-\infty}^{t} h(s)ds$.

***(6 marks)***

(ii) A system, $S$, acts as a differentiator,

$$S(f(t)) = \frac{df}{dt}.$$

Show that $S$ is linear and shift-invariant and find its STF. Show that

$$\frac{d(f * g)}{dt} = \frac{df}{dt} * g = f * \frac{dg}{dt}.$$

***(5 marks)***

(iii) Find the STF for a system with response $g(t)$ for input $f(t)$ where

$$g(t) = \frac{f(t) - f(t-T)}{T}.$$

Show that this system has the same STF as the differentiator as $T \to 0$. Find the impulse response function and the step response function (using the expression given in [i]) writing the latter using $p_a(t)$ notation.

***(10 marks)***

(iv) For the system in (iii) find the response to the input $\cos 2t + 3\sin t$ using the inverse Fourier transform. Check that your result is consistent with direct differentiation in the limit $T \to 0$. ***(4 marks)***

**4** (i) A signal, $f(t)$, with Fourier transform, $F(\omega)$, is sampled with period $T$ to obtain the sampled signal $f_s(t)$. Assuming the Fourier Transform pair $\bar{\delta_T}(t) \leftrightarrow \sigma\bar{\delta_\sigma}(\omega)$, where $\bar{\delta_T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, $\bar{\delta_\sigma}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - n\sigma)$ and $\sigma = 2\pi/T$, prove that

$$f_s(t) \equiv f(t)\bar{\delta_T}(t) \leftrightarrow \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\sigma).$$

If $f(t)$ is $\Omega$-bandlimited and $T = 0.5$ seconds, what is the largest value of $\Omega$ in (rad/s) for which the spectrum consists of non-overlapping copies of $F(\omega)$. ***(5 marks)***

(ii) For an $\Omega$-bandlimited signal, define the Nyquist frequency. Explain, using a clear diagram, how a signal may be reconstructed exactly from a sampled signal using a low-pass filter if sampled at a rate greater than the Nyquist frequency.

Show that this process is equivalent to convolving the sampled signal with a sinc function in the time domain and hence prove the sinc interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\mathrm{sinc}\left(\frac{\sigma}{2}(t - nT)\right)$$

where $T < \pi/\Omega$. ***(9 marks)***

(iii) Find the Nyquist frequency (in Hz) of the signal $f(t) = \mathrm{sinc}^2(\pi t)$. The signal is sampled at half the Nyquist frequency and the samples are used to form a signal $g(t)$ by sinc interpolation. Making use of clear diagrams, find $G(\omega)$ and hence $g(t)$. ***(11 marks)***

## End of Question Paper

**Function Definitions:**

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

**Fourier Transform Pairs:**

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a} p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a} q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a + i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t - t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega - \omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

**Fourier transform:**

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

**Inverse Fourier transform:**

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

**Duality theorem:** If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

**Scaling:** If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \frac{1}{|a|}F(\omega/a)$.

**Translation:** If $f(t) \longleftrightarrow F(\omega)$ then $f(t - t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

**Frequency Shift:** If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega - \omega_0)$

**Fourier Series:** If $f_T(t)$ is periodic with period $T$ then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\mathrm{e}^{-in\sigma t}\ \mathrm{d}t$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\cos n\sigma t\ \mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\sin n\sigma t\ \mathrm{d}t$$

**Parseval's Theorem:** If $V$ is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for $V$ and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

**Plancherel's Theorem:** If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\ \mathrm{d}\omega$$

**Energy Theorem:** If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

**Convolution Theorem:** If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\ \mathrm{d}s \longleftrightarrow F(\omega)G(\omega)$$

**Product Theorem:** If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$

`[fsheet.tex: 24th August 2011, Rev. 1.1]`