SCHOOL OF MATHEMATICS AND STATISTICS

MAS420 Signal Processing

Attempt ALL questions.

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1 (i) In the Hilbert space of finite power signals of period $T$, under the inner product
\[ (f, g) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)g^*(t) \, dt, \]
prove that the set $\phi_n(t) = e^{int}; -\infty < t < \infty$, where $\sigma = 2\pi/T$, form an orthogonal set (both the orthogonality and unit norm conditions must be proved). \(\text{(6 marks)}\)

(ii) Sketch the periodic function, $f(t)$, of period 2, for $-3 \leq t < 3$, where
\[ f(t) = \begin{cases} 1 & -1 < t \leq 0 \\ -1 & 0 < t \leq 1 \end{cases} \]
Find the complex Fourier coefficients for this signal. \(\text{(8 marks)}\)

(iii) Use Parseval’s theorem and your result in (ii) to derive the equality
\[ \frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \]
\(\text{(5 marks)}\)

(iv) The signal is passed over a link which will not pass frequencies greater than $2\pi$ rad s$^{-1}$ but passes other frequencies unchanged. The received signal is $g(t)$. Find an expression for $g(t)$ as a sine/cosine series and show that during transmission approximately 19% of the power has been lost. \(\text{(6 marks)}\)
(i) Making use of clear sketches find \( g(t) = p_a(t) * p_b(t) \), where \( 0 < a < b \), and 
\( p_c(t) \) is the rectangular pulse of width \( 2c \), and calculate its energy \( E(g) \).  
(9 marks)

(ii) Prove the convolution theorem: if \( f(t) \) and \( g(t) \) are signals with Fourier Transforms \( F(\omega) \) and \( G(\omega) \) respectively, then

\[
(f * g)(t) \leftrightarrow F(\omega)G(\omega). 
\]

Use the theorem to find the Fourier Transform of the signal \( g(t) \) of part (i).  
(7 marks)

(iii) Prove the Plancherel theorem: if \( f(t) \) and \( g(t) \) are signals with Fourier transforms \( F(\omega) \) and \( G(\omega) \) respectively, then

\[
(f, g) = \frac{1}{2\pi} (F, G). 
\]

Deduce that

\[
\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = 2\pi E(f), 
\]

where \( E(f) \) is the energy in the signal \( f(t) \).

Use this and the results of part (i) (with a suitable choice of \( a \) and \( b \)) to show that

\[
\int_{-\infty}^{\infty} \text{sinc}^4(\omega) d\omega = \frac{2\pi}{3}. 
\]

(9 marks)
3. (i) A signal $x(t)$ with Fourier Transform $X(\omega)$ is input to a system with system transfer function (STF) $H(\omega)$. Define the system transfer function. Write down the expression for the Fourier Transform, $Y(\omega)$, of the signal, $y(t)$, that is output from the system.

Given that $h(t)$, the impulse response function, is the inverse Fourier transform of the STF and the step response function, $s(t)$, is the response of a system to the unit step signal, $U(t)$, prove that $s(t) = \int_{-\infty}^{t} h(s)ds$. 

(6 marks)

(ii) A system, $S$, acts as a differentiator,

$$S(f(t)) = \frac{df}{dt}.$$

Show that $S$ is linear and shift-invariant and find its STF. Show that

$$\frac{d(f \ast g)}{dt} = \frac{df}{dt} \ast g = f \ast \frac{dg}{dt}.$$

(5 marks)

(iii) Find the STF for a system with response $g(t)$ for input $f(t)$ where

$$g(t) = \frac{f(t) - f(t - T)}{T}.$$

Show that this system has the same STF as the differentiator as $T \to 0$. Find the impulse response function and the step response function (using the expression given in [ii]) writing the latter using $p_a(t)$ notation.

(10 marks)

(iv) For the system in (iii) find the response to the input $\cos 2t + 3 \sin t$ using the inverse Fourier transform. Check that your result is consistent with direct differentiation in the limit $T \to 0$.

(4 marks)
(i) A signal, \( f(t) \), with Fourier transform, \( F(\omega) \), is sampled with period \( T \) to obtain the sampled signal \( f_s(t) \). Assuming the Fourier Transform pair \( \delta_T(t) \leftrightarrow \sigma \delta_\sigma(\omega) \), where \( \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \), \( \delta_\sigma(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - n\sigma) \) and \( \sigma = 2\pi/T \), prove that

\[
f_s(t) \equiv f(t)\delta_T(t) \leftrightarrow \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\sigma).
\]

If \( f(t) \) is \( \Omega \)-bandlimited and \( T = 0.5 \) seconds, what is the largest value of \( \Omega \) in (rad/s) for which the spectrum consists of non-overlapping copies of \( F(\omega) \). (5 marks)

(ii) For an \( \Omega \)-bandlimited signal, define the Nyquist frequency. Explain, using a clear diagram, how a signal may be reconstructed exactly from a sampled signal using a low-pass filter if sampled at a rate greater than the Nyquist frequency.

Show that this process is equivalent to convolving the sampled signal with a sinc function in the time domain and hence prove the sinc interpolation formula

\[
f(t) = \sum_{n=-\infty}^{\infty} f(nT) \text{sinc} \left( \frac{\sigma}{2} (t - nT) \right)
\]

where \( T < \pi/\Omega \). (9 marks)

(iii) Find the Nyquist frequency (in Hz) of the signal \( f(t) = \text{sinc}^2(\pi t) \). The signal is sampled at half the Nyquist frequency and the samples are used to form a signal \( g(t) \) by sinc interpolation. Making use of clear diagrams, find \( G(\omega) \) and hence \( g(t) \). (11 marks)

End of Question Paper
Function Definitions:
Rectangular pulse:
\[ p_a(t) = \begin{cases} 
1 & |t| \leq a \\
0 & |t| > a 
\end{cases} \]

Triangular pulse:
\[ q_a(t) = \begin{cases} 
1 - |t|/a & |t| \leq a \\
0 & |t| > a 
\end{cases} \]

Step function:
\[ U(t) = \begin{cases} 
1 & t \geq 0 \\
0 & t < 0 
\end{cases} \]

Fourier Transform Pairs:
\[ p_a(t) \leftrightarrow 2a \text{sinc}(a\omega) \]
\[ q_a(t) \leftrightarrow a \text{sinc}^2(a\omega/2) \]
\[ \text{sinc}(at) \leftrightarrow \frac{\pi}{a} p_a(\omega) \]
\[ \text{sinc}^2(at) \leftrightarrow \frac{\pi}{a} q_{2a}(\omega) \]
\[ e^{-at}U(t) \leftrightarrow \frac{1}{a + i\omega} \]
\[ \delta(t) \leftrightarrow 1 \]
\[ \delta(t-t_0) \leftrightarrow e^{-i\omega t_0} \]
\[ 1 \leftrightarrow 2\pi \delta(\omega) \]
\[ e^{i\omega t} \leftrightarrow 2\pi \delta(\omega - \omega_0) \]
\[ e^{-t^2/2\sigma^2} \leftrightarrow \sigma \sqrt{2\pi} e^{-\sigma^2\omega^2/2} \]

Fourier transform:
\[ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \]

Inverse Fourier transform:
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega \]

Duality theorem: If \( f(t) \leftrightarrow F(\omega) \) then \( F(t) \leftrightarrow 2\pi f(-\omega) \)

Scaling: If \( f(t) \leftrightarrow F(\omega) \) then \( f(at) \leftrightarrow \frac{1}{|a|} F(\omega/a) \).

Translation: If \( f(t) \leftrightarrow F(\omega) \) then \( f(t-t_0) \leftrightarrow e^{-i\omega t_0} F(\omega) \).

Frequency Shift: If \( f(t) \leftrightarrow F(\omega) \) then \( e^{i\omega t} f(t) \leftrightarrow F(\omega - \omega_0) \)
**Fourier Series:** If $f_T(t)$ is periodic with period $T$ then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\sigma t}$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-in\sigma t} \, dt$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \cos n\sigma t \, dt$$
$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \sin n\sigma t \, dt$$

**Parseval’s Theorem:** If $V$ is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for $V$ and $f = \sum_n c_n \phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

**Plancherel’s Theorem:** If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t) g^*(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G^*(\omega) \, d\omega$$

**Energy Theorem:** If $f(t) \leftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

**Convolution Theorem:** If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$ then

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s) g(t-s) \, ds \leftrightarrow F(\omega) G(\omega)$$

**Product Theorem:** If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$ then

$$f(t)g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega).$$