



The
University
Of
Sheffield.

MAS430

SCHOOL OF MATHEMATICS AND STATISTICS

**Autumn Semester
2016–17**

Analytic Number Theory

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

Note that the questions do not carry equal marks.

**Please leave this exam paper on your desk
Do not remove it from the hall**

Registration number from U-Card (9 digits)
to be completed by student

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- 1 (i) Show that there are infinitely many primes of the form $6k - 1$. *(4 marks)*
- (ii) Let q be a prime and let $\Phi(x) = x^{q-1} + x^{q-2} + \dots + x^2 + x + 1$ be the q -th cyclotomic polynomial.
- (a) Let us call a prime p " Φ -divisor" if there is an integer a such that p divides $\Phi(a)$. Show that there are infinitely many Φ -divisors. *(3 marks)*
- (b) Let $p > q$ be a prime and assume that p divides $\Phi(a)$ for some integer a . Show that p divides $a^q - 1$, and deduce that the residue class of a in $(\mathbb{Z}/p\mathbb{Z})^*$ has order q . *(5 marks)*
- (c) Deduce that there are infinitely many primes of the forms $qn + 1$. *(3 marks)*
- (iii) State Bertrand's Postulate. *(1 mark)*
- (a) Use Bertrand's Postulate to show that for every $n \geq 1$, there is an odd m such that $0 < m < 2n$ and $2n + m$ is prime. *(3 marks)*
- (b) Use Bertrand's Postulate to show that

$$n \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right)$$

is never an integer for any integer $n \geq 3$. *(6 marks)*

- 2 (i) Give a formula for the highest power of a prime p that divides $n!$ and calculate the number of zeros at the end of $1016!$. *(4 marks)*
- (ii) For $x \geq 1$, let $N_n(x)$ denote the number of integers $1 \leq k \leq x$ that are divisible *only* by the first n primes. Show that

$$N_n(x) \leq 2^n \sqrt{x}.$$

Deduce that there are infinitely many primes. *(6 marks)*

- (iii) Consider the following sum

$$S := \sum_{p \text{ prime}} \frac{1}{p}.$$

Using the inequality in Part (ii), show that S is divergent. Deduce that there are infinitely many primes. *(9 marks)*

3 (i) (a) Define $\pi(x)$ and state the Prime Number Theorem (PNT). *(2 marks)*

(b) For $x > 0$, define $\pi_2(x)$ to be the number of primes p such that $p^2 \leq x$. Assuming PNT, show that

$$\pi_2(x) \sim \frac{2\sqrt{x}}{\ln x}.$$

(4 marks)

(c) For $0 < a < b$, calculate, assuming PNT, the limit

$$\lim_{x \rightarrow \infty} \frac{\pi_2(ax)}{\pi_2(bx)}.$$

Deduce that we can find two primes p and q such that

$$a < \frac{p^2}{q^2} < b.$$

(9 marks)

(ii) Define the Riemann zeta function $\zeta(s)$ and write down the Euler product for $\zeta(s)$, indicating in what region of the complex plane they are valid.

(4 marks)

(iii) Define $\lambda(n)$ as

$$\lambda(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^{e_1 + \dots + e_k}, & \text{if } n = p_1^{e_1} \dots p_k^{e_k} \text{ with prime } p_i \text{'s.} \end{cases}$$

(a) Prove that λ is completely multiplicative. *(3 marks)*

(b) Prove that

$$\sum_{d|n} \lambda(n) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise} \end{cases}$$

(Hint: Use Part (a).) *(3 marks)*

(c) Prove that

$$D(s, \lambda) = \frac{\zeta(2s)}{\zeta(s)}.$$

(Hint: Use Part (a).) *(5 marks)*

- 4 (i) For $k \geq 0$, the Bernoulli polynomials $B_k(x)$ are defined by the expression

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

For $k \geq 1$, show the following:

(a) $\frac{d}{dx}(B_k(x)) = kB_{k-1}(x),$ **(4 marks)**

(b) $\int_0^1 B_k(x)dx = 0.$ **(3 marks)**

- (ii) This question asks you to illustrate the proof of Dirichlet's Theorem in a specific case.

(a) List the characters of $(\mathbb{Z}/12\mathbb{Z})^*$. **(5 marks)**

(b) Prove that $L(1, \chi) \neq 0$ for each non-trivial character χ on your list. **(6 marks)**

(c) Using the character table, show that for a prime p , we have

$$\sum_{\chi} \chi(\bar{7})^{-1} \chi(\bar{p}) = \begin{cases} 4 & \text{if } p \equiv 7 \pmod{12} \\ 0 & \text{otherwise} \end{cases}$$

where the sum runs over the characters of $(\mathbb{Z}/12\mathbb{Z})^*$. **(2 marks)**

(d) Prove that there are infinitely many primes congruent to 7 modulo 12.

(You may assume that for any character χ , the sum $\sum_{p \neq 2,3} \sum_{n=2}^{\infty} \frac{\chi(p)}{np^{ns}}$ converges to a finite limit as $s \rightarrow 1$.) **(6 marks)**

End of Question Paper