The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016-17**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) State the Subfield Criterion. ***(4 marks)***

(ii) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{2} : \ a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + bi\sqrt{3} + ci : \ a, b, c \in \mathbb{Q}\}$. ***(3 marks)***

(iii) Consider the subfield $L = \mathbb{Q}(\sqrt{3}, i\sqrt{3})$ of $\mathbb{C}$.

(a) Find $[L : \mathbb{Q}]$. Justify your answer and give a $\mathbb{Q}$-basis of L. ***(7 marks)***

(b) Prove that $L = \mathbb{Q}(i, \frac{1}{\sqrt{3}})$. ***(3 marks)***

(c) Find $\frac{1 - i\sqrt{3}}{1 + i\sqrt{3}}$. The answer should be given in terms of the basis of (a). ***(3 marks)***

2 (i) (a) State Eisenstein's Irreducibility Criterion. *(2 marks)*

(b) Prove Eisenstein's Irreducibility Criterion. *(8 marks)*

(c) Show that the polynomial $3x^4 - 10x^3 - 35x^2 + 35$ is irreducible in $\mathbb{Q}[x]$. *(2 marks)*

(ii) (a) State the "backwards version" of Eisenstein's Irreducibility Criterion. *(3 marks)*

(b) Prove the "backwards version" of Eisenstein's Irreducibility Criterion. *(7 marks)*

(c) Show that the polynomial $f(x) = 2x^3 + 6x^2 + 3$ is irreducible in $\mathbb{Q}[x]$ by using the "backwards version" of Eisenstein's Irreducibility Criterion (or otherwise). *(3 marks)*

3 (i) Let $f \in \mathbb{Z}[x]$. Prove that the polynomial f is reducible in $\mathbb{Z}[x]$ if and only if it is reducible in $\mathbb{Q}[x]$. *(7 marks)*

(ii) State the Degrees Theorem. *(3 marks)*

(iii) Let $L = \mathbb{Q}(a, b)$ where $a = \sqrt[p]{3}$, $b = \sqrt[q]{-2}$, p and q are distinct prime numbers.

(a) Find a $\mathbb{Q}$-basis of the field L. *(10 marks)*

(b) Express the element

$$\frac{3a + 2b}{a^{\frac{p+1}{2}} - b^{\frac{q+1}{2}}}$$

of L as a $\mathbb{Q}$-linear combination of the basis elements from (a). *(5 marks)*

4 Let $K \subseteq L$ be fields.

(i) Define the group $G(L/K)$. *(2 marks)*

(ii) Find the group $G(\mathbf{Q}(\sqrt{3})/\mathbf{Q})$. *(8 marks)*

(iii) Find the group $G(\mathbb{F}_7(\sqrt{\overline{3}})/\mathbb{F}_7)$ where $\mathbb{F}_7 = \{\overline{0}, \overline{1}, \ldots, \overline{6}\}$ is the field that contains 7 elements and $\overline{3} \in \mathbb{F}_7$. *(10 marks)*

(iv) Let F be a finite field of characteristic p. Prove that the field contains p^n elements where n is a natural number. *(5 marks)*

End of Question Paper