



The
University
Of
Sheffield.

MAS6352

SCHOOL OF MATHEMATICS AND STATISTICS

**Spring Semester
2016–2017**

MAS6352 Measure and Probability

2 hours 30 minutes

Full marks may be obtained by complete answers to three questions. All answers will be marked, but credit will be given only for the best three answers. Total marks 99.

**Please leave this exam paper on your desk
Do not remove it from the hall**

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1 (i) Given two *probability* measures P_1 and P_2 on a measurable space (S, Σ) and a number $0 \leq c \leq 1$, define $P(A) = cP_1(A) + (1 - c)P_2(A)$ for all $A \in \Sigma$. Show that P is also a probability measure on (S, Σ) . **(4 marks)**

(ii) (a) Let (S, Σ, m) be a measure space with m a finite measure, that is $m(S) < \infty$. Let $A_n, n \geq 1$ be an increasing sequence of subsets of S , that is $A_n \subseteq A_{n+1}$ for all $n \geq 1$. Define $A := \cup_{n=1}^{\infty} A_n$. Show that

$$m(A) = \lim_{n \rightarrow \infty} m(A_n).$$

(HINT: Write $\cup_{n=1}^{\infty} A_n$ as a disjoint union.) **(4 marks)**

(b) Let (S, Σ, m) be a measure space with m a finite measure, that is $m(S) < \infty$. Let $B_n, n \geq 1$ be a decreasing sequence of subsets of S , that is $B_{n+1} \subseteq B_n$ for all $n \geq 1$. Define $B := \cap_{n=1}^{\infty} B_n$. Show that

$$m(B) = \lim_{n \rightarrow \infty} m(B_n).$$

(HINT: Consider $A_n = S - B_n$ and use part (a).) **(5 marks)**

(c) Give a counterexample to show that the above identity does not hold in general when the sets B_n are decreasing and m is an *infinite* measure. **(2 marks)**

(iii) Consider the set $S = \{1, 2, 3, 4, 5\}$. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Write down the smallest σ -algebra Σ of S which contains A and B . **(5 marks)**

(iv) Recall that the Borel σ -algebra $\mathcal{B}(\mathbb{R})$ is the smallest σ -algebra of \mathbb{R} that contains open intervals (a, b) , $-\infty \leq a < b \leq \infty$. Show that $\mathcal{B}(\mathbb{R})$ contains sets of the form $[a, b]$ and $\{a\}$, where $-\infty < a < b < \infty$. **(5 marks)**

(v) For any set $A \subseteq \mathbb{R}$ define $-A := \{-x : x \in A\}$. Consider the collection

$$\mathcal{C} = \{A \in \mathcal{B}(\mathbb{R}) : -A \in \mathcal{B}(\mathbb{R})\}.$$

(a) Show that \mathcal{C} is a σ -algebra. **(5 marks)**

(b) Show that $\mathcal{C} = \mathcal{B}(\mathbb{R})$. **(3 marks)**

- 2 (i) Consider the sequence $a_n, n \geq 1$ given by

$$a_n = \begin{cases} 1 - \frac{1}{n}, & \text{if } n \text{ is odd} \\ -1 + \frac{1}{n}, & \text{if } n \text{ is even} \end{cases}$$

- (a) Compute $\limsup_{n \rightarrow \infty} a_n$ and $\liminf_{n \rightarrow \infty} a_n$. *(3 marks)*
- (b) Does $\lim_{n \rightarrow \infty} a_n a_{n+1}$ exist? If so, what is the limit? *(2 marks)*
- (ii) Let (S, Σ) be a measurable space and let $f, g : S \rightarrow \mathbb{R}$ be two measurable functions.
- (a) Show that $f + \mathbf{1}$ is a measurable function, where $\mathbf{1} : S \rightarrow \mathbb{R}$ is the function which is identically one, $\mathbf{1}(s) = 1$ for all $s \in S$. *(3 marks)*
- (b) Show that $\{f > g\} \in \Sigma$.
(HINT: If $f(x) > g(x)$ then there must be a rational point between $f(x)$ and $g(x)$) *(4 marks)*
- (iii) Give an example of a measurable space (S, Σ) and a function $f : S \rightarrow \mathbb{R}$ such that $|f|$ is measurable but f is not. *(4 marks)*
- (iv) Let S be a set with $A \subseteq S$ and consider the σ -algebra $\Sigma = \{\emptyset, A, A^c, S\}$. Show that a function $f : S \rightarrow \mathbb{R}$ is measurable if and only if it is constant on A and also constant on A^c .
(HINT: For one direction consider $f^{-1}(\{a\})$ for $a \in \mathbb{R}$.) *(5 marks)*
- (v) Let (S, Σ, m) be a measure space and $f : S \rightarrow \mathbb{R}$ be an integrable function.
- (a) Suppose $m(f \neq 0) = 0$. Show that $\int_S f dm = 0$.
(HINT: Write $f = f \cdot \mathbf{1}_{\{f=0\}} + f \cdot \mathbf{1}_{\{f \neq 0\}}$) *(4 marks)*
- (b) Compute $\int_0^1 \mathbf{1}_{\mathbb{Q}}(x) dx$. *(1 mark)*
- (c) Suppose $\int_S f dm = 0$. Is it true that $m(f \neq 0) = 0$? Prove it if it is true or else provide a counterexample if it is false. *(3 marks)*
- (vi) Consider $([0, 2], \mathcal{B}([0, 2]), \lambda)$ where λ is the Lebesgue measure on $[0, 2]$. Let $f : [0, 2] \rightarrow \mathbb{R}$ be a nonnegative integrable function such that $f(x) \leq 3$ for $x \leq 1$ and $\int_1^2 f(x) dx = 2$. Show that *(4 marks)*

$$\lambda(x : f(x) \geq 4) \leq \frac{1}{2}.$$

- 3 (i) State the Dominated Convergence theorem and use it to find the limit

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} [1 - e^{-|x|/n}] \cdot e^{-|x|} dx.$$

(5 marks)

- (ii) Let (S, Σ, m) be a measure space. The *Monotone Convergence theorem* states that for any monotonic increasing sequence of non negative measurable functions f_n from S to \mathbb{R} we have

$$\int_S \lim_{n \rightarrow \infty} f_n dm = \lim_{n \rightarrow \infty} \int_S f_n dm.$$

Give a counterexample to show that the above identity does not hold if the functions f_n are monotonic decreasing. (3 marks)

- (iii) Let (S, Σ, m) be a measure space. Let (f_n) be a sequence of non-negative measurable functions for which $f_n \leq f$ for all $n \in \mathbb{N}$ where f is integrable. Prove that

$$\limsup_{n \rightarrow \infty} \int_S f_n dm \leq \int_S \limsup_{n \rightarrow \infty} f_n dm.$$

(HINT: Apply Fatou's lemma to $f - f_n$.) (5 marks)

- (iv) Let (Ω, \mathcal{F}, P) be a probability space and let X be a random variable that takes positive integer values.

(a) Deduce that $X = \sum_{i=1}^{\infty} \mathbf{1}_{\{X \geq i\}}$.

(HINT: Consider the event $\{X(\omega) = k\}$) (3 marks)

(b) Show that $\mathbb{E}(X) = \sum_{i=1}^{\infty} P(X \geq i)$. (3 marks)

- (v) Prove that in any infinite sequence of independent (fair) coin tosses, the pattern *HTHHT* appears infinitely often, where *H* represents heads and *T* represents tails. (4 marks)

- (vi) Consider the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda)$ where λ is the uniform measure on $[0, 1]$. Show that the random variables $X_n = n \cdot \mathbf{1}_{(0, n^{-1})}$ converge almost surely to $X \equiv 0$ but X_n does not converge in mean square to X . (4 marks)

- (vii) Let (Ω, \mathcal{F}, P) be a probability space and let X_1, X_2, \dots be a sequence of i.i.d. random variables with mean 0 and variance 1. Let $S_n = X_1 + X_2 + \dots + X_n$, $n \geq 1$ and consider the event $A_n = \{S_n \in [1, 2]\}$. Show that

$$\mathbb{E} [(S_{n+1}^2 - (n+1)) \cdot \mathbf{1}_{A_n}] = \mathbb{E} [(S_n^2 - n) \cdot \mathbf{1}_{A_n}]$$

(6 marks)

- 4 (i) Let (S_1, Σ_1, m_1) and (S_2, Σ_2, m_2) be measure spaces. Recall that for $E \subseteq S_1 \times S_2$ and $x \in S_1$ the x -slice of E is

$$E_x := \{y \in S_2 : (x, y) \in E\}.$$

Let $E, F \subseteq S_1 \times S_2$ and $x \in S_1$. Show that

- (a) $(E \cap F)_x = E_x \cap F_x$. *(3 marks)*
 (b) $(E^c)_x = (E_x)^c$ *(3 marks)*
 (c) $(\cup_{n=1}^{\infty} E_n)_x = \cup_{n=1}^{\infty} (E_n)_x$ where $E_n, n \geq 1$ is a sequence of subsets of $S_1 \times S_2$. *(3 marks)*

- (ii) State the version of Fubini's theorem for nonnegative measurable functions. *(4 marks)*

- (iii) Let (Ω, \mathcal{F}, P) be a probability space and let X be a nonnegative random variable with $0 \leq X \leq 1$. Consider the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda)$. Consider the product space $\Omega \times [0, 1]$ with product σ -algebra and product probability $P \times \lambda$.

- (a) Show that the set G is in the product σ -algebra, where

$$G = \{(\omega, y) : y \leq X(\omega)\}.$$

(HINT: Consider G^c and note that if $X(\omega) < y$ then there must be a rational number between $X(\omega)$ and y) *(5 marks)*

- (b) Show that $P \times \lambda(G) = E(X)$. *(5 marks)*

- (iv) (a) Let (S, Σ) be a measurable space and let m_1 and m_2 be two finite measures on it with the property $m_1(S) = m_2(S)$. Show that the collection

$$\mathcal{C} := \{A \in \Sigma : m_1(A) = m_2(A)\}$$

is a λ -system. *(5 marks)*

- (b) Show that the Lebesgue measure is the only measure m on the Borel sets of the interval $[0, 1]$ with the property that for all subintervals J , $m(J) = \text{length of } J$.
 (HINT: Use Dynkin's $\pi - \lambda$ theorem) *(5 marks)*

End of Question Paper