The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

MAS6450 Waves and Magnetohydrodynamics **2 hours**

Answer all four questions. Formulae are on the last page.

1 (i) Consider the equation in the form

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)-\frac{1}{\mu_0}\left(\mathbf{B}\cdot\nabla\right)\mathbf{B}=0.$$

For a magnetic field $\mathbf{B}=B_0\tanh\frac{x}{L}\hat{y}$, where $\hat{y}$ is the unit vector along the y-direction in the Cartesian coordinate system (x, y, z), show that

$$p+\frac{B^2}{2\mu_0}$$

is constant. In the above, p is gas pressure, μ_0 is the magnetic permeability in a vacuum and L, a typical length scale. ***(3 marks)***

(ii) In cylindrical coordinates $(\hat{r}, \hat{\theta}, \hat{z})$, consider a purely azimuthal magnetic field $\mathbf{B}=\frac{B_0}{r}\,\hat{\theta}$, where B_0 is a constant. Calculate the current density, vector potential and magnetic pressure gradient. ***(10 marks)***

(iii) A static radially symmetric corona with temperature

$$T(r)=T_0\left(\frac{r_0}{r}\right)^{2/7}$$

is in equilibrium under a balance between a pressure gradient and gravity, $\left(\frac{MG\rho}{r^2}\right)\hat{r}$ where M is the mass and G is the universal gravitational constant. T_0, p_0 and ρ_0 are the temperature, pressure and density respectively at a reference distance $r=r_0$. Find the pressure $p(r)$ and density $\rho(r)$. Show that according to this model the pressure at large distances is much greater than the interstellar pressure of $p_0/10^{15}$. Comment on this last fact.

[Hint: use $\frac{MG}{r_0RT_0}=15$; R is a gas constant.] ***(12 marks)***

2 (i) Consider a magnetic field

$$\mathbf{B} = \left(\frac{\partial \psi}{\partial z}, B_y(x, z), -\frac{\partial \psi}{\partial x} \right),$$

where $\psi = \psi(x, z)$.

(a) Show that $\nabla \cdot \mathbf{B} = 0$. ***(2 marks)***

(b) Show that $\mathbf{B} \cdot \nabla \psi = 0$ and that projections of field lines in the xz-plane are given by ψ=constant. ***(6 marks)***

(c) Show that if the Lorentz force

$$\mathbf{J} \times \mathbf{B} = \mathbf{0},$$

then

$$B_y = B_y(\psi)$$

and ψ satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + B_y \frac{dB_y}{d\psi} = 0.$$

(12 marks)

(ii) Calculate the approximate timescale (in years) for the decay of the interstellar magnetic field given the parameters: length scale $L = 3 \times 10^{18}$ cm and magnetic diffusivity $\eta = 3.6 \times 10^6$ cm^2 s^{-1}. ***(5 marks)***

3 (i) Verify that a solution of the form $B(x, t) = f(t) e^{-x^2/(4\eta t)}$ satisfies the diffusion equation

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2},$$

if

$$\frac{df}{dt} = -\frac{f}{2t}.$$

Here η is the magnetic diffusivity. Integrate to find $f(t)$ and therefore $B(x, t)$ if $B(0, t_0) = B_0$ [i.e., if $f(t_0) = B_0$]. ***(13 marks)***

(ii) Verify that the magnetic field $\mathbf{B}(x, y, z) = (x, y, 6y - 2z)$ is solenoidal. Calculate the equations of the field lines in yz-plane.

(8 marks)

(iii) Sketch the magnetic fields for $\mathbf{B} = (x, 0, 1)e^{-z}$ on the xz-plane with arrows indicating the direction of the field. ***(4 marks)***

4 (i) Write down the linearised equations of MHD for adiabatic perturbations about a uniform state at rest.
[Hint: consider an inviscid, perfectly conducting, incompressible fluid with no gravity].

(5 marks)

Deduce what they become for perturbations that are proportional to $exp[i(\omega t - \mathbf{k} \cdot \mathbf{r})]$.

(5 marks)

Derive the dispersion relation for Alfvén waves from these equations. ***(5 marks)***

(ii) Find how the non-dimensional flow given by $\mathbf{v} = (\sin z, \cos z, 0)$ in the Eulerian representation deforms the non-dimensional magnetic field given by $\mathbf{B} = (0, 0, 1)$ at $t = 0$. ***(10 marks)***

 Question 4 continued on next page

4 (continued)

Formulae Sheet

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

	u	v	w	f	g	h
cartesian	x	y	z	1	1	1
spherical	r	θ	ϕ	1	r	$r\sin\theta$
cylindrical	r	ϕ	z	1	r	1

$$\nabla \cdot \mathbf{V} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghV_u) + \frac{\partial}{\partial v}(fhV_v) + \frac{\partial}{\partial w}(fgV_w)\right]$$

$$\nabla \times \mathbf{V} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(hV_w) - \frac{\partial}{\partial w}(gV_v)\right]\hat{u} \;+\; \frac{1}{fh}\left[\frac{\partial}{\partial w}(fV_u) - \frac{\partial}{\partial u}(hV_w)\right]\hat{v} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gV_v) - \frac{\partial}{\partial v}(fV_u)\right]\hat{w}$$

vector identity:

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

End of Question Paper