The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016–17**

Analytic Number Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.
Note that the questions do not carry equal marks.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Show that there are infinitely many primes of the form $6k-1$. ***(4 marks)***

(ii) Let q be a prime and let $\Phi(x) = x^{q-1} + x^{q-2} + \ldots + x^2 + x + 1$ be the q-th cyclotomic polynomial.

(a) Let us call a prime p "Φ-divisor" if there is an integer a such that p divides $\Phi(a)$. Show that there are infinitely many Φ-divisors. ***(3 marks)***

(b) Let $p > q$ be a prime and assume that p divides $\Phi(a)$ for some integer a. Show that p divides $a^q - 1$, and deduce that the residue class of a in $(\mathbb{Z}/p\mathbb{Z})^\star$ has order q. ***(5 marks)***

(c) Deduce that there are infinitely many primes of the forms $qn + 1$. ***(3 marks)***

(iii) State Bertrand's Postulate. ***(1 mark)***

(a) Use Bertrand's Postulate to show that for every $n \geq 1$, there is an odd m such that $0 < m < 2n$ and $2n + m$ is prime. ***(3 marks)***

(b) Use Bertrand's Postulate to show that

$$n\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

is never an integer for any integer $n \geq 3$. ***(6 marks)***

2 (i) Give a formula for the highest power of a prime p that divides $n!$ and calculate the number of zeros at the end of $1016!$. ***(4 marks)***

(ii) For $x \geq 1$, let $N_n(x)$ denote the number of integers $1 \leq k \leq x$ that are divisible *only* by the first n primes. Show that

$$N_n(x) \leq 2^n\sqrt{x}.$$

Deduce that there are infinitely many primes. ***(6 marks)***

(iii) Consider the following sum

$$S := \sum_{p \text{ prime}} \frac{1}{p}.$$

Using the inequality in Part (ii), show that S is divergent. Deduce that there are infinitely many primes. ***(9 marks)***

3 (i) (a) Define $\pi(x)$ and state the Prime Number Theorem (PNT).

(2 marks)

(b) For $x > 0$, define $\pi_2(x)$ to be the number of primes p such that $p^2 \leq x$. Assuming PNT, show that

$$\pi_2(x) \sim \frac{2\sqrt{x}}{\ln x}.$$

(4 marks)

(c) For $0 < a < b$, calculate, assuming PNT, the limit

$$\lim_{x \to \infty} \frac{\pi_2(ax)}{\pi_2(bx)}.$$

Deduce that we can find two primes p and q such that

$$a < \frac{p^2}{q^2} < b.$$

(9 marks)

(ii) Define the Riemann zeta function $\zeta(s)$ and write down the Euler product for $\zeta(s)$, indicating in what region of the complex plane they are valid.

(4 marks)

(iii) Define $\lambda(n)$ as

$$\lambda(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^{e_1 + \ldots + e_k}, & \text{if } n = p_1^{e_1} \cdots p_k^{e_k} \text{ with prime } p_i\text{'s}. \end{cases}$$

(a) Prove that λ is completely multiplicative. *(3 marks)*

(b) Prove that

$$\sum_{d|n} \lambda(n) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise} \end{cases}$$

(Hint: Use Part (a).) *(3 marks)*

(c) Prove that

$$D(s, \lambda) = \frac{\zeta(2s)}{\zeta(s)}.$$

(Hint: Use Part (a).) *(5 marks)*

4 (i) For $k \geq 0$, the Bernouilli polynomials $B_k(x)$ are defined by the expression

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

For $k \geq 1$, show the following:

(a) $\frac{d}{dx}(B_k(x)) = kB_{k-1}(x)$, ***(4 marks)***

(b) $\int_0^1 B_k(x)dx = 0.$ ***(3 marks)***

(ii) This question asks you to illustrate the proof of Dirichlet's Theorem in a specific case.

(a) List the characters of $(\mathbb{Z}/12\mathbb{Z})^{\star}$. ***(5 marks)***

(b) Prove that $L(1, \chi) \neq 0$ for each non-trivial character χ on your list. ***(6 marks)***

(c) Using the character table, show that for a prime p, we have

$$\sum_{\chi} \chi(\overline{7})^{-1} \chi(\overline{p}) = \begin{cases} 4 & \text{if } p \equiv 7 \mod 12 \\ 0 & \text{otherwise} \end{cases}$$

where the sum runs over the characters of $(\mathbb{Z}/12\mathbb{Z})^{\star}$. ***(2 marks)***

(d) Prove that there are infinitely many primes congruent to 7 modulo 12.

(You may assume that for any character χ, the sum $\sum_{p \neq 2,3} \sum_{n=2}^{\infty} \frac{\chi(p)}{np^{ns}}$ converges to a finite limit as $s \to 1$.) ***(6 marks)***

End of Question Paper