The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Mathematics Core II **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.
There are 55 marks available on the paper.

1 Consider the three equations in four variables:

$$\begin{aligned} x - y + 2z - t &= 2 \\ 3x - y + 4z - 7t &= 8 \\ x + 3y - 2z - 9t &= 6 \end{aligned}$$

(i) Write this system in augmented matrix form.

(ii) Perform a complete reduction on this augmented matrix.

(iii) Give all solutions to the system in parametric form. ***(4 marks)***

2 Find all values of α such that the vector $(-2, \alpha, 1)$ is a linear combination of the vectors $(1, 2, 3)$, $(1, -1, 2)$ and $(1, 8, 5)$. ***(3 marks)***

3 (i) If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the *determinant* of M.

(ii) Let A and B be $n \times n$ matrices. Write down a formula that expresses $\det(AB)$ in terms of $\det A$ and $\det B$.

(iii) Verify explicitly that your formula holds for the matrices $A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$. ***(4 marks)***

4 (i) Define the terms *eigenvector* and *eigenvalue* for an $n \times n$ matrix A.

(ii) Show that the characteristic polynomial of the matrix $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ has a double root, and find all eigenvectors with this eigenvalue. ***(5 marks)***

5 Let $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$. Find M such that $M^{-1}AM$ is diagonal. ***(4 marks)***

6 Let A be an invertible $n \times n$ matrix. Which of the following statements is *not* necessarily true? Give a counterexample.

A One can row reduce A to the identity matrix.

B One of the eigenvalues of A is 1.

C $\det A \neq 0$.

D None of the eigenvalues of A is 0.

(2 marks)

7 Give the equation of the tangent plane and normal vector to the surface $z = 3x^2 + 2xy - y^2$ at the point $(1, 1, 4)$. ***(4 marks)***

8 Let $f(x, y) = \sqrt{1 + xy}$. Evaluate f at the point $(3, 2)$. Using partial derivatives, approximate the change δf when we move to the nearby point $(3.1, 2.1)$. To how many decimal places does this agree with the actual value of δf? ***(6 marks)***

9 Suppose that $u = u(x, y)$ and $v = v(x, y)$ are two functions of variables x and y.

(i) Define the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$.

(ii) If $u = x^2 + y + 1$ and $v = x^4 + 2x^2y + y^2 - x^2 - y$, show that $\dfrac{\partial(u, v)}{\partial(x, y)} = 0$. By completing the square, find an explicit polynomial h with $v - h(u) = 0$.

(iii) Conversely, given any u and v related by $v = h(u)$ for some function h, write down expressions for $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ using the Chain Rule. Deduce that $\dfrac{\partial(u, v)}{\partial(x, y)} = 0$. ***(6 marks)***

10 Find all the stationary points of the function

$$f(x, y) = x^3 + 2y^3 - 3x - 6y.$$

Further, determine the nature of each stationary point. ***(7 marks)***

11 (i) Give the formula for the area of the surface of revolution obtained by rotating the graph of a function $f(x)$ around the x-axis over an interval $[a, b]$.

(ii) Calculate the surface area of the paraboloid obtained by rotating the curve $y = x^{\frac{1}{2}}$ about the x-axis over the interval $[0, 2]$. ***(4 marks)***

12 Find $\displaystyle\iint_D e^{x^2+y^2}\, dA$, where D is the domain in the upper right quadrant of the xy-plane bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, the y-axis and the line $y = x$. ***(6 marks)***

End of Question Paper