The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Vectors and Mechanics **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets. The total number of marks available is 60. Use $g = 9.8\ ms^{-2}$ when needed.

1 Let $ABCD$ be an arbitrary quadrilateral. The mid-points of AB, BC, CD and DA are P, Q, R and S, respectively. Start by finding the position vectors of P, Q, R and S, then prove that $PQRS$ is a parallelogram. ***(5 marks)***

2 Relative to the origin O, the position vectors of points F and G are $\overrightarrow{OF} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\overrightarrow{OG} = -\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, respectively. Find a unit vector $\overrightarrow{OH}$ in the direction of $\overrightarrow{FG}$. ***(2 marks)***

3 Let $\boldsymbol{F} = (\alpha\boldsymbol{i} + 10\boldsymbol{k})$N be a force where the $\boldsymbol{i}$ and $\boldsymbol{k}$ components are along the horizontal and vertical directions, respectively. Given that $|\boldsymbol{F}| = 20$N, find

(i) the value of α,

(ii) the angle in radians between $\boldsymbol{F}$ and the horizontal. ***(2 marks)***

4 The angle between vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\pi/3$ radians. Also $|\boldsymbol{u}| = |\boldsymbol{v}| = 3$. Find

(i) $|\boldsymbol{u} - \boldsymbol{v}|$,

(ii) $|\boldsymbol{u} + \boldsymbol{v}|$. ***(3 marks)***

5 Let $\boldsymbol{c} = 2\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{d} = \beta\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$. Given that $\boldsymbol{c}$ is perpendicular to $\boldsymbol{d}$, find β. ***(2 marks)***

6 Let $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$. Find

(i) $\boldsymbol{a} \times \boldsymbol{b}$,

(ii) $(\boldsymbol{a} + 2\boldsymbol{b}) \times (2\boldsymbol{a} - \boldsymbol{b})$. ***(3 marks)***

7 A line is defined by the vector equation $\boldsymbol{r} \times (\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) = 2\boldsymbol{i} - 2\boldsymbol{k}$ and a plane is given by $\boldsymbol{r} \cdot (2\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}) = 4$. Find

(i) where the line and plane intersect,

(ii) the angle in radians between the line and the plane to 2 significant figures.

(6 marks)

8 The equation of motion for a charged particle moving with a velocity $\boldsymbol{v}(t)$ in a magnetic field $\boldsymbol{B}$ is given by

$$m\frac{d\boldsymbol{v}}{dt} = q(\boldsymbol{v} \times \boldsymbol{B}),$$

where the constant scalars m and q are the particle's mass and electric charge, respectively.

(i) Calculate $\boldsymbol{v} \cdot \dfrac{d\boldsymbol{v}}{dt}$.

(ii) By considering the time derivative of $\boldsymbol{v} \cdot \boldsymbol{v}$ show that the speed of the charged particle is always constant in the presence of a magnetic field. ***(3 marks)***

9 A bob of mass m is suspended by a light spring of natural length a and stiffness k. Find the subsequent displacement of the body if initially
(i) the bob is pulled down a distance $\dfrac{a}{4}$ from its position of equilibrium and released,
(ii) the bob is given a downward speed v_0 from its position of equilibrium.

(7 marks)

10 A car of mass 1000 kg climbs a hill which makes an angle α to the horizontal, where $\sin\alpha = \dfrac{1}{8}$. The maximum speed of the car is 50 m s^{-1}. If the power of the car is 125 kW, what is the frictional force acting on the car? After lubrication, this frictional force is halved. What is the new maximum speed of the car up the hill? ***(7 marks)***

11 (i) A bead is moving on a smooth helical wire such that, at time t, its position vector relative to an origin O is given by

$$\mathbf{r}(t) = 2\cos\theta\ \boldsymbol{i} + 2\sin\theta\ \boldsymbol{j} - 2\theta\ \boldsymbol{k},$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are perpendicular unit vectors, with $\boldsymbol{k}$ pointing vertically upwards, with $\theta = \theta(t)$. Find the velocity and acceleration of the bead at time t in terms of θ and derivatives of θ. Draw a sketch showing the forces acting on the bead. ***(5 marks)***

(ii) A conical pendulum has a string of length 2m. If the pendulum makes 1 revolution per second, find the angle in radians the string makes with the vertical. ***(4 marks)***

12 A particle P moves in a circle of radius a, centre O. The line $\overrightarrow{OP}$ makes an angle θ with the positive x-axis. Let $\mathbf{e_r}$ be a unit vector in the direction $\overrightarrow{OP}$, and $\mathbf{e}_\theta$ be a unit vector in the direction of increasing θ.

(i) Write down expressions for $\mathbf{e_r}$ and $\mathbf{e}_\theta$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$, the unit vectors in the x and y directions respectively. Hence deduce that

$$\frac{d}{d\theta}\mathbf{e_r} = \mathbf{e}_\theta, \quad \frac{d}{d\theta}\mathbf{e}_\theta = -\mathbf{e_r}.$$

(4 marks)

(ii) Hence show that the speed of P is $a|\dot{\theta}|$, and that

$$\dot{\boldsymbol{v}} \;=\; a\ddot{\theta}\mathbf{e}_\theta - a\dot{\theta}^2\mathbf{e_r},$$

where $\boldsymbol{v}$ is the velocity of P. *(7 marks)*

End of Question Paper