The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2017–2018**

Numbers and Groups — **2 hours**

Answer ***all questions****.*
You should justify your answers carefully unless the question states otherwise.

1 (i) (a) State Fermat's Little Theorem. ***(2 marks)***

(b) What is the remainder left by dividing 47^{765432} by 101? ***(3 marks)***

(ii) Find all solutions in positive integers m and n to

$$42m + 76n = 900.$$

(5 marks)

2 (i) Let $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers.

We define the following functions:

- The function $a : \mathbb{Z} \to \mathbb{Z}$ defined by $a(x) = x^{10}$.
- The function $b : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by $b(x) = x^{10}$.
- The function $c : \mathbb{R} \to \mathbb{R}$ defined by $c(x) = x^{10}$.
- The function $d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $d(x) = x^{10}$.

Make a copy of the following table:

	injective	surjective	bijective
a			
b			
c			
d			

Into every empty cell of the table, place either a tick (✓) to indicate that the function does have that property, or a cross ($\times$) to indicate that the function does not have that property. ***(5 marks)***

(ii) Show carefully by induction that $(2n)! \leq 4^n (n!)^2$ for all positive integers n. ***(5 marks)***

3 (i) Prove carefully that there is no rational number x satisfying $x^2 = 5$.
(3 marks)

(ii) For each one of the following three statements, state whether it is true or false. When it is false, give a counterexample.

(a) Whenever a is rational and b is rational, then $a + b$ is rational.

(b) Whenever a is irrational and b is irrational, then $a + b$ is irrational.

(c) Whenever a is irrational and b is rational, then $a + b$ is irrational.
(3 marks)

(iii) Prove directly from the definition of convergence that the sequence $a_1, a_2, \ldots$ defined by $a_n = e^{-\sqrt{n}}$ converges. ***(4 marks)***

4 (i) In S_6, let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 6 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \beta = (1\ 3\ 4)(3\ 6\ 2)(2\ 4\ 1\ 6).$$

Find the orders of α and β, and determine whether they are odd or even.
(4 marks)

(ii) Let $n \geq 2$. For each statement below, determine whether it is true or false.

(Note: you must justify your answers to get the marks, using appropriate reasoning or counter-examples where necessary.)

(a) For all $\alpha, \beta, \gamma \in S_n$ we have $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

(b) If k is a positive integer such that $\alpha^k = \mathrm{id}$, then α has order k.

(c) If $\alpha \in S_n$ then α^2 is even.

(d) There is no $\alpha \in S_3$ with $\alpha^2 = (1\ 2)$.

(e) There is no $\alpha \in S_3$ with $\alpha^2 = (1\ 2\ 3)$.

(f) There is no element of S_6 of order 7. ***(6 marks)***

5 (i) Let G be any group and let $g \in G$. Show that $H = \{g^k : k \in \mathbb{Z}\}$ is a subgroup of G. ***(3 marks)***

(ii) Let $G = O_2$, the group of symmetries of the circle. For each of the below, clearly justify why it is not a subgroup of G.

(a) $H_1 = \{f \in O_2 : f = \text{ref}_\theta \text{ for some } \theta \in \mathbb{R}\}$

(b) $H_2 = \{f \in O_2 : f = \text{rot}_\theta \text{ for some } 0 \leq \theta \leq \pi\}$ ***(2 marks)***

(iii) Let G be a group and let $g \in G$ have order $2n$ for some positive integer n. Suppose that $a \in G$ is such that $a^2 = g$.

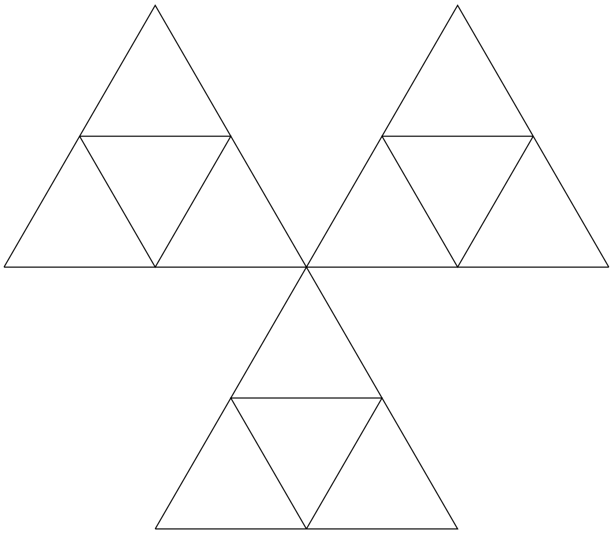
(a) Show that if k is a positive integer with $a^k = e$, then $g^k = e$. ***(1 mark)***

(b) Show that $a^{2n} \neq e$. ***(1 mark)***

(c) Let the order of a be k. Show that $k = 4n$. ***(3 marks)***

6 (i) Define an relation on the set of real numbers by $a \sim b \iff a - b \in \mathbb{Q}$ for all $a, b \in \mathbb{R}$. Is $\sim$ an equivalence relation? Justify your answer. ***(3 marks)***

(ii) A toy boomerang, which can be turned over, is to be made by gluing together 12 pieces of coloured plastic, each in the shape of an equilateral triangle, in the pattern shown below.



Find the number of essentially different ways in which the toy can be made with 6 red pieces and 6 blue pieces, clearly explaining your reasoning and any formulas you use. ***(5 marks)***

(iii) Let $S = \{z \in \mathbb{C} : z^n = 1 \text{ for some positive integer } n\}$. Is S a countable set? ***(2 marks)***

End of Question Paper