



The
University
Of
Sheffield.

MAS152

SCHOOL OF MATHEMATICS AND STATISTICS

**Spring Semester
2016–2017**

**MAS152 Essential Mathematical Skills and
Techniques**

3 hours

*Attempt **ALL** the questions.*

Each question in Section A carries 3 marks,

each question in Section B carries 8 marks.

All solutions should be justified in full. Calculators should be relied upon for simple steps like basic arithmetic and plugging numbers into elementary functions.

Section A

A1 Let $f(x) = \frac{x}{x^2 + 1}$. Find the stationary points and sketch a graph of $y = f(x)$.

A2 Let $f(x) = \frac{\ln x}{x}$. State the domain and the range of $y = f(x)$.

A3 Let $f(x, y) = e^x \sin y$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial x \partial y}$.

A4 Using l'Hospital's Rule, or otherwise, evaluate $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$.

A5 Find all complex numbers z for which $|z - i| = |z + i|$.

A6 Let $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (1, -2\lambda, 1)$. Find a value of λ such that \mathbf{a} and \mathbf{b} are perpendicular. Also, find a value of λ such that \mathbf{a} and \mathbf{b} are parallel.

A7 Find the definite integral $\int_0^1 xe^{-x} dx$.

A8 Find the indefinite integral $\int x \ln x dx$.

A9 For $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, find A^{-1} . Without doing any further matrix multiplication, indicate why $AA^T = A^T A = I$ holds.

A10 Consider the following system of equations

$$\begin{cases} x + 2y = 0, \\ \lambda x + 3y = 0, \end{cases}$$

and let $A = \begin{pmatrix} 1 & 2 \\ \lambda & 3 \end{pmatrix}$. Considering the determinant of A , find the value of λ for the above system to have a unique solution. Also, find the value of λ for the system to have infinitely many solutions.

A11 Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}.$$

A12 Find the general solution of the differential equation

$$\frac{dy}{dx} + y = x.$$

Section B

- B1** (i) Let $g(y) = \sin^{-1} y + y\sqrt{1-y^2}$. Show that $\frac{dg}{dy} = 2\sqrt{1-y^2}$.
- (ii) Consider the function $f(x) = g\left(\frac{x}{a}\right)$ where a is a fixed constant and g is the function introduced in (i). Calculate $\frac{df}{dx}$ to show that

$$\int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C,$$

where C is constants.

- B2** Find all the cube roots of $z = \sqrt{3} + i$ and plot them on the Argand diagram.

- B3** The position vector $\mathbf{r}(t)$ of a particle is given by

$$\mathbf{r}(t) = (t^2 + t, t^3 - t, t^3 - t^2).$$

- (i) When $t = 1$, find the velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ and acceleration $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$.
- (ii) Again, when $t = 1$, find a unit vector \mathbf{t} in the direction of \mathbf{v} and the component of \mathbf{a} in the direction of \mathbf{v} .

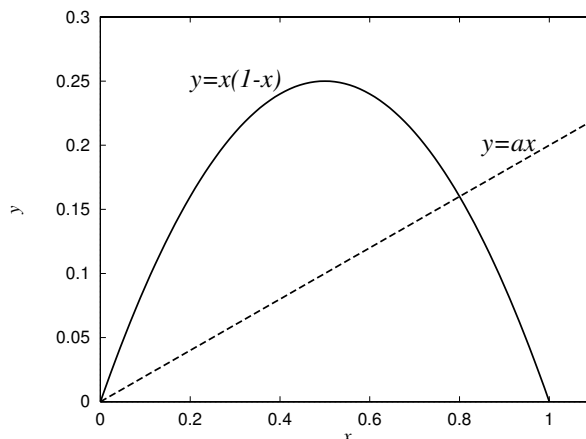
- B4** Consider the differential equation

$$\frac{d^2y}{dx^2} - (a+1)\frac{dy}{dx} + ay = 0.$$

- (i) Find the general solution when $a \neq 1$.
- (ii) Find the general solution when $a = 1$.
- (iii) Using l'Hôpital rule, find

$$\lim_{a \rightarrow 1} \frac{e^{ax} - e^x}{a - 1},$$

thereby relating the solutions in (i) and (ii).



B5 (i) Show that

$$\int_0^\alpha x(\alpha - x) dx = \frac{\alpha^3}{6},$$

for $\alpha > 0$.

(ii) Consider an area bounded by a graph of $y = x(1 - x)$ and the x -axis in the above figure. Find a constant a ($0 < a < 1$) such that $y = ax$ bisects the area.

B6 Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix}$. Find all the eigenvalues of A and corresponding eigenvectors.

B7 Show that

$$\int_0^{\pi/2} \ln \sin x dx = -\frac{\pi}{2} \ln 2$$

by making use of either

$$\text{Hint (1)} \quad \int_0^{\pi/2} \ln \sin x dx = \int_0^{\pi/2} \ln \cos x dx = \frac{1}{2} \int_0^{\pi/2} \ln \frac{\sin 2x}{2} dx,$$

or

$$\text{Hint (2)} \quad \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \dots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}} \quad (n \geq 2).$$

There is no need to prove the Hints.

B8 Using Gauss elimination, solve the following system of linear equations

$$\begin{cases} x + 2y + 7z = 3, \\ -2x + 5y + 4z = 3, \\ -5x + 6y - 3z = 1. \end{cases}$$

End of Question Paper

MAS140/151/152/156/161 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

Trigonometry

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

$$2 \sin^2 x = 1 - \cos 2x$$

$$2 \cos^2 x = 1 + \cos 2x$$

$$2 \sin x \cos x = \sin 2x$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$a \cos x + b \sin x = R \cos(x - \alpha)$$

$$\text{where } R = \sqrt{a^2 + b^2},$$

$$\cos \alpha = \frac{a}{R} \quad \text{and} \quad \sin \alpha = \frac{b}{R}$$

$$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2 \tan(x/2)}{1 + \tan^2(x/2)}$$

$$\tan x = \frac{2 \tan(x/2)}{1 - \tan^2(x/2)}$$

Hyperbolic Functions

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$2 \cosh^2 x = 1 + \cosh 2x$$

$$2 \sinh^2 x = \cosh 2x - 1$$

$$2 \sinh x \cosh x = \sinh 2x$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \text{ all } x$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \quad |x| < 1$$

Series

Sum of an arithmetic series:

$$\frac{\text{first term} + \text{last term}}{2} \times (\text{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}$

Binomial theorem: $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \binom{n}{r}x^r + \dots$

$$\text{where } \binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

If n is a positive integer then the series terminates and the result is true for all x , otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left. \begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \\ \exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned} \right\} \text{valid for all } x$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (-1 < x \leq 1)$$

Differentiation

<u>Function</u>	<u>Derivative</u>	<u>Function</u>	<u>Derivative</u>
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, x < 1$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}, x < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\frac{1}{\cosh^2 x}$	$\coth x$	$-\frac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}, x > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, x < 1$		
$\coth^{-1} x$	$\frac{1}{1-x^2}, x > 1$		

Integration

In the following table the constants of integration have been omitted.

$$\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln |x|$$

$$\int e^x dx = e^x$$

$$\int a^x dx = \frac{a^x}{\ln a} \quad (a > 0, a \neq 1)$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \sec^2 x dx = \tan x$$

$$\int \operatorname{cosec}^2 x dx = -\cot x$$

$$\int \sinh x dx = \cosh x$$

$$\int \cosh x dx = \sinh x$$

$$\int \operatorname{sech}^2 x dx = \tanh x$$

$$\int \operatorname{cosech}^2 x dx = -\operatorname{coth} x$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \quad (|x| < a)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \quad (|x| > a)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| \quad \left(= \frac{1}{a} \tanh^{-1} \frac{x}{a} \quad \text{if } |x| < a \right)$$

$$\int \operatorname{cosec} x dx = \ln \tan \left(\frac{x}{2} \right) \quad \text{or} \quad \ln (\operatorname{cosec} x - \cot x)$$

$$\int \sec x dx = \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \quad \text{or} \quad \ln (\sec x + \tan x)$$

$$\int \operatorname{cosech} x dx = \ln \tanh \left(\frac{x}{2} \right)$$

Integration by parts

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

Variable substitution in definite integral

If $x = \varphi(t)$ is a monotonic function in the interval $[\alpha, \beta]$ and $a = \varphi(\alpha)$, $b = \varphi(\beta)$, then

$$\int_a^b f(x) dx = \int_\alpha^\beta f(\varphi(t)) \varphi'(t) dt$$

Variable substitution for a rational function of $\sin x$ and $\cos x$

Let $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\frac{dx}{dt} = \frac{2}{1+t^2}$.

Table of Laplace transforms

<u>Function $f(t)$</u>	<u>Laplace transform $F(s)$</u>
t^n	$\frac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \dots$)
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
$e^{at} f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$