



The
University
Of
Sheffield.

MAS153/MAS159

SCHOOL OF MATHEMATICS AND STATISTICS

**Spring Semester
2016–2017**

**Mathematics (Materials)
Mathematics For Chemists**

3 hours

All questions are compulsory. The marks awarded to each question or section of question are shown in italics. Total Marks 100.

1 Given

$$3 \cos(2\theta) = 4 \cos(\theta) - 1.$$

Find θ .

(6 marks)

2 Find the inverse of the following:

$$y = e^x - e^{2x} + 5.$$

(6 marks)

3 Factorise $3x^2 + 4x - 15$.

(2 marks)

4 Show that

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}.$$

(5 marks)

5 State the domain of the function

$$y(x) = \begin{cases} \frac{x^2}{2x-1} & -1 \leq x \leq 2 \\ \frac{2x}{2x-1} & x > 2. \end{cases} \quad (2 \text{ marks})$$

6 (i) For the circle $x^2 + 2x + y^2 - 4y - 11 = 0$, find the equations of the tangent which meets the circle at $(-1, 6)$ and the normal at that point. *(6 marks)*

(ii) Find where this tangent intersects the circle with centre $(-5, 9)$ and radius 5. *(6 marks)*

7 If $y = e^{x^2}$, find $\frac{d^2y}{dx^2}$ in terms of y . *(4 marks)*

8 (i) Showing your working clearly, find the coefficient of x^4 in the expansion of $(1 + x/3)^{17}$. *(2 marks)*

(ii) Use the binomial theorem to evaluate

$$\lim_{x \rightarrow \infty} \sqrt{x^2 + 2x - 12} - x,$$

stating the limit placed upon x . *(7 marks)*

9 (i) Find the angle between the vectors $\mathbf{a} = (2, 1, 5)$ and $\mathbf{b} = (3, 7, 9)$, giving your answer in radians to two decimal places. *(3 marks)*

(ii) A plane passes through the points $(2, 1, 5)$, $(3, 7, 9)$ and $(0, 2, -3)$. Find the Cartesian equation of the plane. *(5 marks)*

10 Prove, from the definitions of $\sinh x$ and $\cosh x$, the identity

$$\sinh^2(x) = -(1 - \cosh(2x))/2. \quad \text{span style="float: right;">*(3 marks)*$$

11 (i) If $y = \cosh(2x)$, show that

$$\cosh^{-1}(2x) = \frac{1}{2} \ln |y + \sqrt{y^2 - 1}|. \quad \text{span style="float: right;">*(2 marks)*$$

(ii) If $y = \cosh^{-1}(3x)$, show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1/9}}. \quad \text{span style="float: right;">*(4 marks)*$$

12 Using partial fractions find the integral of

$$\int \frac{x^2 + 3}{(x^2 - 4)(x - 1)} dx. \quad \text{span style="float: right;">*(6 marks)*$$

13 Find the Taylor series (around 2) for e^{-x^2} , as far as the term in x^3 . *(7 marks)*

14 Using De-Moivres theorem (Isolating the real and imaginary functions) show that

$$\cos(4\theta) = 8 \cos^4 \theta - 8 \cos^2 \theta + 1,$$

and

$$\sin(4\theta) = 4 \sin(\theta) \cos(\theta)(1 - 2 \sin^2 \theta).$$

(12 marks)

15 A set of linear equations can be written as

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2 & -3 & 5 \\ 1 & \lambda & 4 \\ 1 & 2\lambda & 7 \end{pmatrix}.$$

Find λ such that there is a non-trivial solution. For the λ found, find the inverse, A^{-1} , of A , and use it to find the values of x , y and z which satisfy the equations.
(12 marks)

End of Question Paper

Formulae Sheet

These results may be quoted without proof, unless proofs are asked for in the question.

Trigonometry

For any angles A and B

$$\sin^2 A + \cos^2 A = 1$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

Coordinate Geometry

The acute angle α between lines with gradients m_1 and m_2 satisfies

$$\tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \quad (m_1 m_2 \neq -1)$$

while the lines are perpendicular if $m_1 m_2 = -1$.

The equation of a circle centre (x_0, y_0) and radius a is $(x - x_0)^2 + (y - y_0)^2 = a^2$.

Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\cosh^2 x + \sinh^2 x = \cosh 2x$$

$$2 \sinh x \cosh x = \sinh 2x$$

$$\cosh^2 x = (1 + \cosh 2x)/2$$

$$\sinh^2 x = -(1 - \cosh 2x)/2$$

Differentiation

Function (y)	Derivative (dy/dx)
x^n	nx^{n-1}
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$
$\tan ax$	$a \sec^2 ax$
e^{ax}	ae^{ax}
$\ln(ax)$	$\frac{1}{x}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

NB. It is assumed that x takes only those values for which the functions are defined.

For u and v functions of x , and with $u' = \frac{du}{dx}$, $v' = \frac{dv}{dx}$,

$$\frac{d}{dx}(uv) = uv' + vu',$$

while

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{vu' - uv'}{v^2}.$$

For $y = y(t)$, $t = t(x)$,

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}.$$

Integration

In the following table the constants of integration have been omitted.

Function $f(x)$	Integral $\int f(x) dx$
x^n	$\frac{x^{n+1}}{n+1} \quad n \neq -1$
ae^{ax}	e^{ax}
$\frac{1}{x}$	$\ln x $
$a \sin ax$	$-\cos ax$
$a \cos ax$	$\sin ax$
$a \tan ax$	$\ln \sec ax $
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{a^2 - x^2}$	$\frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{x^2 + a^2}}$	$\sinh^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1} \left(\frac{x}{a} \right)$
$\frac{f'(x)}{f(x)}$	$\ln f(x) $

Integration by parts

$$\int uV dx = (\text{integral of } V) \times u - \int (\text{integral of } V) \times \frac{du}{dx} dx$$

or

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

Series

Binomial Theorem: $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \binom{n}{r}x^r + \dots$

where $\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$

If n is a positive integer, the series terminates and is convergent for all x .

If n is not a positive integer, the series is infinite and converges for $|x| < 1$.

Taylor expansion of $f(x)$ about $x = a$ is

$$f(a) + (x - a)f^{(1)}(a) + \frac{(x - a)^2}{2!}f^{(2)}(a) + \dots + \frac{(x - a)^n}{n!}f^{(n)}(a) + \dots$$

Maclaurin expansion of $f(x)$ is

$$f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

Alternating Series Test

The series $a_1 - a_2 + a_3 - a_4 + \dots$, where $a_1, a_2, a_3, a_4, \dots$ are all positive, converges if $a_1 > a_2 > a_3 > \dots$ and $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Ratio Test

If the series $a_1 + a_2 + a_3 + a_4 + \dots$ satisfies

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lambda,$$

then

- 1 if $\lambda > 1$, the series diverges,
- 2 if $\lambda < 1$, the series converges.

Vectors

If vectors \mathbf{a} and \mathbf{b} are given in Cartesian component form by $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

the scalar product $\mathbf{a} \cdot \mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

and the vector product $\mathbf{a} \times \mathbf{b}$ is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

If a plane passes through a point with position vector \mathbf{a} , and is normal to the vector \mathbf{n} , then the equation of the plane is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n},$$

where $\mathbf{r} = (x, y, z)$.