The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets. There is a total of 60 marks.

1 (i) Given a group G, what does it mean for G to be abelian? ***(1 mark)***

(ii) Prove that the permutation group S_3 is not abelian. ***(1 mark)***

(iii) Given a group G, and an element $x \in G$, define the conjugacy class $\mathrm{conj}_G(x)$ and the centraliser $\mathrm{cent}_G(x)$. What is the relation between $|G|$, $|\mathrm{cent}_G(x)|$ and $|\mathrm{conj}_G(x)|$? ***(3 marks)***

(iv) Consider the element $\alpha = (1\,2\,3)(4\,5) \in S_5$. Write down an element $\beta \neq \alpha$ in S_5 such that $\beta \in \mathrm{conj}_G(x)$, and an element $\gamma \neq \mathrm{id}$ in S_5 such that $\gamma \in \mathrm{cent}_G(x)$. ***(2 marks)***

(v) Write down a 2-element subgroup of the multiplicative group $\mathbb{R}^\times = \mathbb{R} - \{0\}$. ***(1 mark)***

(vi) Write down a 4-element subgroup of the multiplicative group $\mathbb{C}^\times = \mathbb{C} - \{0\}$. ***(1 mark)***

(vii) Write 13 as a product of irreducibles in the ring of Gaussian integers $\mathbb{Z}[i] := \{a + bi \ : \ a, b \in \mathbb{Z}\}$. ***(1 mark)***

(viii) Is $\mathbb{C}$ a field? (Yes or No.) ***(1 mark)***

(ix) Is $\mathbb{Z}$ a field? (Yes or No.) ***(1 mark)***

(x) Is $\mathbb{Z}/5\mathbb{Z}$ a field? (Yes or No.) ***(1 mark)***

(xi) Is $\mathbb{Z}[x]$ a Euclidean domain? (Yes or No.) ***(1 mark)***

(xii) Is the matrix ring $M_2(\mathbb{R})$ commutative? (Yes or No.) ***(1 mark)***

(xiii) Is $\mathbb{Z}[i]$ an integral domain? (Yes or No.) ***(1 mark)***

(xiv) Write down the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \in M_{2,3}(\mathbb{R})$. ***(1 mark)***

2 (i) What does it mean for a function $f : G \to H$, where G and H are groups, to be a *group homomorphism*? Define the *kernel*, $\ker(f)$, and prove that it is a subgroup of G. You may assume that $f(e_G) = e_H$ (where e_G and e_H are the neutral elements of G and H, respectively), and that $f(g^{-1}) = (f(g))^{-1}$ for all $g \in G$. ***(4 marks)***

(ii) Why can the subgroup $\{\mathrm{id}, (12)\}$ of S_3 not be the kernel of a homomorphism (from S_3 to some other group)? ***(1 mark)***

3 (i) What does it mean for a function $f : R \to S$, where R and S are rings, to be a *ring homomorphism*? ***(2 marks)***

(ii) Consider the evaluation homomorphism $\mathrm{ev}_{(1,2)} : \mathbb{R}[x, y] \to \mathbb{R}$ given by $\mathrm{ev}_{(1,2)}(f(x, y)) = f(1, 2)$. Write down three different elements in $\ker(\mathrm{ev}_{(1,2)})$. ***(2 marks)***

4 Let $\mathbb{F}_3 := \mathbb{Z}/3\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}\}$ be the field with 3 elements, and $\mathbb{F}_3[x]$ the ring of polynomials in one variable with coefficients in $\mathbb{F}_3$.

(i) Explain why $x^2 + x - 1$ is an irreducible element of $\mathbb{F}_3[x]$. (For convenience, I have written "1" in place of "$\overline{1}$".) ***(1 mark)***

(ii) Find $a, b \in \mathbb{F}_3$ such that $x^3 + x - 1 \equiv ax + b \pmod{x^2 + x - 1}$. How many elements does the quotient ring $\mathbb{F}_3[x]/\langle x^2 + x - 1\rangle$ contain? (Consider a and b.) Is it a field? ***(3 marks)***

(iii) Find a non-zero element of the quotient ring $\mathbb{F}_3[x]/\langle x^2 + x + 1\rangle$ (note the change in sign) whose square is zero. ***(1 mark)***

5 (i) What does it mean for a function $f : V \to W$, where V and W are F-vector spaces, F a field, to be a *linear map*? Prove that the kernel, $\ker(f)$, is a subspace of V. (You may assume that $f(0_V) = 0_W$, and that $\lambda 0_W = 0_W$ for any $\lambda \in F$.) ***(4 marks)***

(ii) Is the map $\det : M_2(\mathbb{R}) \to \mathbb{R}$, given by $\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$, linear? Justify your answer. ***(1 mark)***

(iii) By solving a set of linear equations, find a basis for the kernel of the linear map $\ell : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $\ell(\mathbf{x}) := A\mathbf{x}$, where $A := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$. Explain why the image of ℓ must be the whole of $\mathbb{R}^2$. Find distinct $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^4$ such that $\ell(\mathbf{x}_1) = \ell(\mathbf{x}_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ***(5 marks)***

(iv) Let $V = C^\infty(\mathbb{R}, \mathbb{R})$, the $\mathbb{R}$-vector space of real-valued functions, with derivatives of all orders, of a real variable. Let $L(V)$ be the ring of linear operators on V, and consider $D \in L(V)$ defined by $D(y) := \dfrac{dy}{dx}$. By solving a homogeneous second-order differential equation, find a basis for the subspace $\ker(D^2 - 3D + 2)$ of V (where "2" means multiplication by 2, so $2(y) = 2y$). Find also distinct $y_1, y_2 \in V$ such that $(D^2 - 3D + 2)(y_1) = (D^2 - 3D + 2)(y_2) = e^{3x}$. ***(3 marks)***

6 (i) What is the dimension of the $\mathbb{R}$-vector space $M_2(\mathbb{R})$ of 2-by-2 matrices with real entries? ***(1 mark)***

(ii) Let $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ and $W = \text{Span}(S)$. Demonstrate how $\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}$ belongs to the subspace W. ***(1 mark)***

7 Let V be a vector space over $\mathbb{R}$, with an inner product $\langle , \rangle$. Let $T \in L(V)$ be a linear operator (i.e. a linear map from V to V).

(i) What does it mean for T to be *self-adjoint* with respect to $\langle , \rangle$? ***(1 mark)***

(ii) Suppose that T is self-adjoint with respect to $\langle , \rangle$, and that v_1, v_2 are eigenvectors for T, with eigenvalues λ_1, λ_2 respectively. (So $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$, with v_1, v_2 non-zero.) Prove that if $\lambda_1 \neq \lambda_2$ then v_1 and v_2 are orthogonal with respect to $\langle , \rangle$. ***(2 marks)***

(iii) What is the dimension of $\mathbb{C}$ as an $\mathbb{R}$-vector space? ***(1 mark)***

(iv) Consider $\langle , \rangle : \mathbb{C} \to \mathbb{R}$ defined by $\langle z, w \rangle = \text{Re}(z\,\overline{w})$ for all $z, w \in \mathbb{C}$. Prove that $(\mathbb{C}, \langle , \rangle)$ is a real inner product space. ***(4 marks)***

(v) Consider the map $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = \overline{z}$ for all $z \in \mathbb{C}$. Prove that T is an $\mathbb{R}$-linear map, and that it is self-adjoint with respect to the above inner product. ***(3 marks)***

(vi) With respect to the $\mathbb{R}$-basis $\{1, i\}$ of $\mathbb{C}$, what is the matrix representing the above $\mathbb{R}$-linear operator T? Find two eigenvectors for T with distinct eigenvalues, and check directly that they are orthogonal with respect to $\langle , \rangle$, in accord with what you proved in (ii). ***(3 marks)***

End of Question Paper