The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **2016–2017**

Analysis I **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

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1 (i) Let A be a non–empty set of real numbers. Using precise mathematical language, state what it means for A to be (a) bounded above, (b) bounded below. Give an example of a set of real numbers that is bounded above, but not below. **(3 marks)**

(ii) If A is a non–empty set of real numbers that is bounded above, prove that for any $\epsilon > 0$, there exists $a \in A$ so that

$$a > \sup(A) - \epsilon,$$

where $\sup(A)$ is the least upper bound of the set A.

[Hint: Use a proof by contradiction.] **(3 marks)**

(iii) Using precise mathematical language, state what it means for a sequence of real numbers (a_n) to be *monotonic increasing.* **(1 mark)**

(iv) If the sequence (a_n) is monotonic increasing and bounded above, prove that it converges to the limit $\alpha = \sup\{a_n, n \in \mathbb{N}\}$. Your proof should include an explanation of why α exists.

[Hint: Use the result of (ii).] **(5 marks)**

(v) Consider the sequence (a_n) given by

$$a_1 = 0, \quad a_{n+1} = \frac{57a_n + 1}{a_n + 57} \text{ for all } n > 1,$$

(a) Use induction to show that $0 \leq a_n \leq 1$ for all $n \in \mathbb{N}$.

[Hint: Consider $a_{n+1} - 1$.] **(5 marks)**

(b) By considering $a_{n+1} - a_n$, show that (a_n) is monotonic increasing. **(3 marks)**

(c) Explain why $\lim_{n\to\infty} a_n$ exists, and use standard techniques, such as algebra of limits, to find its value. **(5 marks)**

2 (i) Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose domain includes the closed interval $[a, b]$.

(a) What does it mean for f to be *bounded* on $[a, b]$? **(2 marks)**

(b) If f is bounded on $[a, b]$, explain carefully why $\sup_{x \in [a,b]} f(x)$ and $\inf_{x \in [a,b]} f(x)$ exist. **(2 marks)**

(c) What does it mean for f to attain its bounds on $[a, b]$? **(2 marks)**

(d) State a condition on f that will ensure that f is bounded and that it attains its bounds on $[a, b]$ **(1 mark)**

(e) Is it true that every continuous function from (a, b) to $\mathbb{R}$ is bounded? If so, provide the main steps in the reasoning; if not, give a counter–example. **(2 marks)**

(ii) Consider the functions $f(x) = x^2 \sin(1/x)$ and $g(x) = \cos(1/x)$, each having domain $\mathbb{R} \setminus \{0\}$.

(a) Does the function g have a limit as $x \to 0$? Give reasoning to support your conclusion.

[Hint: You might find it helpful to consider sequences (x_n) and (y_n) where for all $n \in \mathbb{N}, x_n = 1/(2\pi n + a)$ and $y_n = 1/(2\pi n + b)$. You will need to choose suitable numerical values for a and b.] **(4 marks)**

(b) Explain briefly why f is continuous. You may use the standard results that $x \to x^2$ and $x \to \sin(x)$ are continuous on $\mathbb{R}$, and that $x \to 1/x$ is continuous on $\mathbb{R} \setminus \{0\}$. **(2 marks)**

(c) Show that f has a continuous extension f_1 to the whole of $\mathbb{R}$. **(5 marks)**

(d) Deduce that f_1 is differentiable on $\mathbb{R}$, but that its derivative f_1' fails to be continuous at zero.

[Hint: Use the result of (a).] **(5 marks)**

3 (i) (a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define the *lower integral*, $L\int_a^b f(t)\,dt$, the *upper integral*, $U\int_a^b f(t)\,dt$, and say what it means for the function to be *Riemann integrable*. **(3 marks)**

(b) Consider the functions $r : [0, 4] \to \mathbb{R}$ and $s : [0, 4] \to \mathbb{R}$, given by

$$r = \chi_{[0,1)} + e\chi_{[1,2)} + e^8\chi_{[2,3)} + e^{27}\chi_{[3,4]},$$
$$s = e\chi_{[0,1)} + e^8\chi_{[1,2)} + e^{27}\chi_{[2,3)} + e^{64}\chi_{[3,4]}.$$

Calculate

$$\int_0^4 r(t)\,dt, \qquad \text{and} \qquad \int_0^4 s(t)\,dt.$$

(4 marks)

(c) Explain, by stating a result from the course, why the function $f : [0, 4] \to \mathbb{R}$, given by $f(t) = e^{t^3}$, is Riemann integrable. **(1 mark)**

(d) Show that

$$1 + e + e^8 + e^{27} \le \int_0^4 e^{t^3}\,dt \le e + e^8 + e^{27} + e^{64}.$$

(4 marks)

(ii) Consider a sequence of functions, (f_n), where $f_n\colon [a, b] \to \mathbb{R}$.

(a) Define what it means to say that the sequence (f_n) *converges pointwise* to a function $f\colon [a, b] \to \mathbb{R}$. **(1 mark)**

(b) Define what it means to say that the sequence (f_n) *converges uniformly* to a function $f\colon [a, b] \to \mathbb{R}$. **(1 mark)**

(iii) For $n \ge 1$, consider the function $f_n : [0, 1] \to \mathbb{R}$, by

$$f_n(t) = \begin{cases} 1 - (2^n - 1)t, & \text{if } 0 \le t < \dfrac{1}{2^n}, \\ \dfrac{1}{2^n - 1}(1 - t), & \text{if } \dfrac{1}{2^n} \le t \le 1. \end{cases}$$

(a) Sketch the graphs of f_1, f_2 and f_3, labelling the coordinates of any important points. **(3 marks)**

(b) Show that the sequence of functions (f_n) converges pointwise and determine the limit function.

[Hint: Consider the cases $t = 0$ and $0 < t \le 1$ separately. For $0 < t \le 1$, consider N such that $t > 1/2^N$.] **(4 marks)**

(c) Is the function f_n continuous? Does the sequence (f_n) converge uniformly? Justify your answer by stating an appropriate result from the course. **(4 marks)**

4 (i) (a) Use the ratio test to show that the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges.

(3 marks)

(b) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$.

(2 marks)

(ii) Use the Weierstrass M-test to show that one can define a function $f : [0, 2\pi] \to \mathbb{R}$ by

$$f(t) = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$$

and that this function is continuous. **(5 marks)**

(iii) Consider a sequence $(\mathbf{a}_n)$ in $\mathbb{R}^2$, where $\mathbf{a}_n = (x_n, y_n)$. We say that the sequence is bounded if there is a real number $C \geq 0$ such that $|\mathbf{a}_n| \leq C$ for all n. Prove that if $(\mathbf{a}_n)$ is a bounded sequence in $\mathbb{R}^2$ then it has a convergent subsequence.

You may assume the result that a bounded sequence of real numbers has a convergent subsequence. **(5 marks)**

(iv) (a) Let $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$. Define the *open ball* with radius r around $\mathbf{x}$. **(1 mark)**

(b) Define what it means for a subset $U \subseteq \mathbb{R}^k$ to be an *open set.* **(1 mark)**

(c) Let $U, V \subseteq \mathbb{R}^k$ be open sets. Prove that $U \cap V$ is also an open set in $\mathbb{R}^k$. **(4 marks)**

(v) Which of the following subsets of $\mathbb{R}^2$ are open in $\mathbb{R}^2$? Justify your answers.

(a)

$$\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}.$$

(b)

$$\{(x, y) \in \mathbb{R}^2 \,|\, x > 0, y > 0\}.$$

(4 marks)

End of Question Paper