1 (i) Sketch the phase line for the following ordinary differential equation (ODE) for \(-1 \leq u \leq 3\)

\[
\frac{du}{dt} = u(1 - u)(u - 2).
\]

State all the equilibrium points for \(u \in \mathbb{R}\), and say which are stable and which are unstable. (3 marks)

(ii) Consider the following system of ODEs

\[
\frac{dx}{dt} = 2x - 2x^2 - xy,
\]

\[
\frac{dy}{dt} = 2y - 2y^2 - 3xy.
\]

Find the equilibrium points where \(x, y \geq 0\). (3 marks)

Write down the Jacobian of the system in Equations (1-2). (2 marks)

Determine the nature (e.g. spiral, node, centre etc.) and stability (i.e. stable or unstable) of the equilibrium points. (5 marks)

Sketch the nullclines of Equations (1-2). (3 marks)

On a separate diagram, sketch the phase portrait for the system for \(x, y \geq 0\). Include sufficiently many trajectories such that the long-term behaviour of the system from any starting-point is qualitatively clear. (5 marks)

(iii) Sketch the direction field for the following ODE, for \(0 < u < 3\) and \(0 \leq t \leq 2\pi\):

\[
\frac{du}{dt} = u \left( 2 + \frac{\cos(t)}{2} - u \right).
\]

Draw an example solution curve on your direction field plot. (1 mark)
Let $\lambda \in \mathbb{R}$ be a constant. For each of the three cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, write down the general solution to the following ODE
\[ y'' + \lambda y = 0. \] (3 marks)

Now suppose we are given the boundary conditions $y(0) = 0$ and $y'(2\pi) = 0$. Find the values of $\lambda$ for which there is a solution to Equation (3) subject to these boundary conditions. (Note: these should form an infinite set of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$) (6 marks)

Write down the eigenfunction $y_n(x)$ associated to each eigenvalue $\lambda_n$. (1 mark)

(ii) Show that $y = x^{-1/2}$ is a solution to the equation $2x^2y'' + xy' - y = 0$. (2 marks)

Use Reduction of Order to find another (linearly independent) solution. (5 marks)

(iii) Consider the following equation
\[ y'' - 2xy = 0, \]
with $y(0) = 1$ and $y'(0) = 1$. If we search for power series solutions of the type
\[ y(x) = \sum_{n=0}^{\infty} a_n x^n, \]
what are the values of $a_n$ for $n = 0, 1, 2, 3, 4$? (6 marks)

(iv) Show that $x = 0$ is a regular singular point of the following equation
\[ x^2 y'' - 2y = 0. \] (2 marks)
Let \( u(x, y) = F(x)H(y) \) be a separable solution of Laplace’s equation in two dimensions

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,
\]

for \( 0 < x < L, \ 0 < y < L \) (where \( L > 0 \) is a positive constant).

Show that the functions \( F(x) \) and \( H(y) \) satisfy the following ordinary differential equations:

\[
\frac{d^2 F}{dx^2} - \alpha F(x) = 0, \quad \frac{d^2 H}{dy^2} + \alpha H(y) = 0,
\]

where \( \alpha \) is an arbitrary constant. \( \text{(3 marks)} \)

If, in addition, the function \( u(x, y) \) is subject to the boundary conditions

\[
u(0, y) = 0, \quad u(L, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = 0,
\]

write down the boundary conditions that must be satisfied by the functions \( F(x) \) and \( H(y) \). \( \text{(1 mark)} \)

Show that, if \( \alpha \geq 0 \), then the only separable solution of Laplace’s equation \( (4) \) subject to the boundary conditions \( (5) \) is the trivial solution \( u(x, y) \equiv 0 \). \( \text{(6 marks)} \)

If \( \alpha < 0 \), find all nontrivial separable solutions of Laplace’s equation \( (4) \) subject to the boundary conditions \( (5) \). \( \text{(6 marks)} \)

Show that the principle of superposition applies to solutions of Laplace’s equation \( (4) \) subject to the boundary conditions \( (5) \). \( \text{(2 marks)} \)

Hence show that the general solution of Laplace’s equation \( (4) \) subject to the boundary conditions \( (5) \) takes the form

\[
u(x, y) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) \cosh \left( \frac{n\pi y}{L} \right),
\]

where \( B_n, \ n = 1, 2, \ldots, \) are arbitrary constants. \( \text{(1 mark)} \)

Find the solution of Laplace’s equation \( (4) \) subject to the homogeneous boundary conditions \( (5) \) and the inhomogeneous boundary condition

\[
u(x, L) = \frac{u_0}{L} x,
\]

where \( u_0 > 0 \) is a positive constant.

You may use the fact that if a function \( f(x) \) defined on an interval \( 0 < x < L \) has the Fourier sine series

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right)
\]

then the coefficients \( b_n \) are given by the integrals

\[
b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx.
\]

\( \text{(6 marks)} \)
4 (i) Find the general solution \( w(\xi, \eta) \) of the first order partial differential equation (PDE)
\[
\frac{\partial w}{\partial \xi} - \frac{3}{\xi} w = 0.
\]
\((3 \text{ marks})\)

(ii) Find the characteristics of the first order linear PDE
\[
x^2 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 3xu.
\]
\((6)\)
Find a change of variables \((x, t) \to (\xi(x, t), \eta(x, t))\) under which the PDE (6) transforms to a PDE of the form
\[
F(\xi, \eta) \frac{\partial w}{\partial \xi} + H(\xi, \eta)w = J(\xi, \eta)
\]
where \(w(\xi, \eta) = u(x, t)\) and \(F(\xi, \eta), H(\xi, \eta)\) and \(J(\xi, \eta)\) are functions to be determined. \((5 \text{ marks})\)
Hence use your solution to part (i) to find the general solution of the PDE (6) valid for \(x > 0\). \((1 \text{ mark})\)

(iii) Show that the second order linear PDE
\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 4x^2 - t^2
\]
\((7)\)
is hyperbolic everywhere in the \((x, t)\)-plane. \((1 \text{ mark})\)
Find the characteristic equations of the PDE (7). \((3 \text{ marks})\)
Hence show that suitable characteristic coordinates for the PDE (7) are
\[
\xi(x, t) = t - 2x, \quad \eta(x, t) = t + 2x.
\]
\((2 \text{ marks})\)
Under the change of variables \((x, t) \to (\xi(x, t), \eta(x, t))\), with \(\xi(x, t)\) and \(\eta(x, t)\) the characteristic coordinates given above, show that the transformed PDE satisfied by \(w(\xi, \eta) = u(x, t)\) is
\[
\frac{\partial^2 w}{\partial \xi \partial \eta} = \frac{1}{16} \xi \eta.
\]
\((8)\)
Find the general solution \(w(\xi, \eta)\) of the PDE (8). \((5 \text{ marks})\)
Hence find the general solution \(u(x, t)\) of the PDE (7). \((1 \text{ mark})\)

End of Question Paper