The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

MAS222 Differential Equations **2.5 hours**

*Attempt **ALL** questions. The allocation of marks is shown in brackets. Total marks 100.*

1 (i) Sketch the phase line for the following ordinary differential equation (ODE) for $-1 \le u \le 3$

$$\frac{du}{dt} = u(1-u)(u-2).$$

State all the equilibrium points for $u \in \mathbb{R}$, and say which are stable and which are unstable. ***(3 marks)***

(ii) Consider the following system of ODEs

$$\frac{dx}{dt} = 2x - 2x^2 - xy, \tag{1}$$

$$\frac{dy}{dt} = 2y - 2y^2 - 3xy. \tag{2}$$

Find the equilibrium points where $x, y \ge 0$. ***(3 marks)***

Write down the Jacobian of the system in Equations (1-2). ***(2 marks)***

Determine the nature (e.g. spiral, node, centre etc.) and stability (i.e. stable or unstable) of the equilibrium points. ***(5 marks)***

Sketch the nullclines of Equations (1-2). ***(3 marks)***

On a **separate diagram**, sketch the phase portrait for the system for $x, y \ge 0$. Include sufficiently many trajectories such that the long-term behaviour of the system from any starting-point is qualitatively clear. ***(5 marks)***

(iii) Sketch the direction field for the following ODE, for $0 < u < 3$ and $0 \le t \le 2\pi$:

$$\frac{du}{dt} = u\left(2 + \frac{\cos(t)}{2} - u\right).$$

(3 marks)

Draw an example solution curve on your direction field plot. ***(1 mark)***

2 (i) Let $\lambda \in \mathbb{R}$ be a constant. For each of the three cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, write down the general solution to the following ODE

$$y'' + \lambda y = 0. \tag{3}$$

(3 marks)

Now suppose we are given the boundary conditions $y(0) = 0$ and $y'(2\pi) = 0$. Find the values of λ for which there is a solution to Equation (3) subject to these boundary conditions. (Note: these should form an infinite set of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \ldots$.) ***(6 marks)***

Write down the eigenfunction $y_n(x)$ associated to each eigenvalue λ_n. ***(1 mark)***

(ii) Show that $y = x^{-1/2}$ is a solution to the equation $2x^2y'' + xy' - y = 0$. ***(2 marks)***

Use Reduction of Order to find another (linearly independent) solution. ***(5 marks)***

(iii) Consider the following equation

$$y'' - 2xy = 0,$$

with $y(0) = 1$ and $y'(0) = 1$. If we search for power series solutions of the type

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

what are the values of a_n for $n = 0, 1, 2, 3, 4$? ***(6 marks)***

(iv) Show that $x = 0$ is a regular singular point of the following equation

$$x^2y'' - 2y = 0.$$

(2 marks)

3 Let $u(x,y) = F(x)H(y)$ be a separable solution of Laplace's equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (4)$$

for $0 < x < L$, $0 < y < L$ (where $L > 0$ is a positive constant).

Show that the functions $F(x)$ and $H(y)$ satisfy the following ordinary differential equations:

$$\frac{d^2F}{dx^2} - \alpha F(x) = 0, \qquad \frac{d^2H}{dy^2} + \alpha H(y) = 0,$$

where α is an arbitrary constant. ***(3 marks)***

If, in addition, the function $u(x,y)$ is subject to the boundary conditions

$$u(0,y) = 0, \qquad u(L,y) = 0, \qquad \frac{\partial u}{\partial y}(x,0) = 0, \qquad (5)$$

write down the boundary conditions that must be satisfied by the functions $F(x)$ and $H(y)$. ***(1 mark)***

Show that, if $\alpha \geq 0$, then the only separable solution of Laplace's equation (4) subject to the boundary conditions (5) is the trivial solution $u(x,y) \equiv 0$. ***(6 marks)***

If $\alpha < 0$, find all nontrivial separable solutions of Laplace's equation (4) subject to the boundary conditions (5). ***(6 marks)***

Show that the principle of superposition applies to solutions of Laplace's equation (4) subject to the boundary conditions (5). ***(2 marks)***

Hence show that the general solution of Laplace's equation (4) subject to the boundary conditions (5) takes the form

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right),$$

where B_n, $n = 1, 2, \ldots$, are arbitrary constants. ***(1 mark)***

Find the solution of Laplace's equation (4) subject to the homogeneous boundary conditions (5) *and* the inhomogeneous boundary condition

$$u(x,L) = \frac{u_0}{L}x,$$

where $u_0 > 0$ is a positive constant.

You may use the fact that if a function $f(x)$ defined on an interval $0 < x < L$ has the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

then the coefficients b_n are given by the integrals

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

(6 marks)

4 (i) Find the general solution $w(\xi, \eta)$ of the first order partial differential equation (PDE)

$$\frac{\partial w}{\partial \xi} - \frac{3}{\xi} w = 0.$$

(3 marks)

(ii) Find the characteristics of the first order linear PDE

$$x^2 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 3xu. \qquad (6)$$

(2 marks)

Find a change of variables $(x, t) \to (\xi(x, t), \eta(x, t))$ under which the PDE (6) transforms to a PDE of the form

$$F(\xi, \eta) \frac{\partial w}{\partial \xi} + H(\xi, \eta) w = J(\xi, \eta)$$

where $w(\xi, \eta) = u(x, t)$ and $F(\xi, \eta)$, $H(\xi, \eta)$ and $J(\xi, \eta)$ are functions to be determined. ***(5 marks)***

Hence use your solution to part (i) to find the general solution of the PDE (6) valid for $x > 0$. ***(1 mark)***

(iii) Show that the second order linear PDE

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial t^2} = 4x^2 - t^2 \qquad (7)$$

is hyperbolic everywhere in the (x, t)-plane. ***(1 mark)***

Find the characteristic equations of the PDE (7). ***(3 marks)***

Hence show that suitable characteristic coordinates for the PDE (7) are

$$\xi(x, t) = t - 2x, \qquad \eta(x, t) = t + 2x.$$

(2 marks)

Under the change of variables $(x, t) \to (\xi(x, t), \eta(x, t))$, with $\xi(x, t)$ and $\eta(x, t)$ the characteristic coordinates given above, show that the transformed PDE satisfied by $w(\xi, \eta) = u(x, t)$ is

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = \frac{1}{16} \xi \eta. \qquad (8)$$

(5 marks)

Find the general solution $w(\xi, \eta)$ of the PDE (8). ***(2 marks)***

Hence find the general solution $u(x, t)$ of the PDE (7). ***(1 mark)***

End of Question Paper