The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Mathematics (Computational and Numerical Methods) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Taking $x_0 = 1$, use the Newton-Raphson method to find the root of the equation

$$\ln(x) + 3x = 1,$$

correct to 6 decimal places. ***(6 marks)***

(ii) If a root is an inflection point do you expect fast or slow convergence using the Newton-Raphson method and why? ***(1 mark)***

(iii) Factorise the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{pmatrix}$$

into the product $A = LU$. Hence, using this LU factorisation, calculate the determinant of A^{-1}. ***(5 marks)***

(iv) Use the Gauss-Seidel iteration method to find an approximate solution to the following system of equations

$$\begin{aligned} -2x_1 + 6x_2 + x_3 &= 9 \\ -x_1 + x_2 + 7x_3 &= -6 \\ 4x_1 - x_2 - x_3 &= 3. \end{aligned}$$

If necessary, first rearrange these equations to ensure convergence. Then starting with the column vector $[0, 0, 0]^T$ compute 3 successive iterations giving your *final* answer to an accuracy of 3 decimal places. ***(6 marks)***

 Question 1 continued on next page

1 (continued)

(v) Matrix B has the inverse

$$B^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}.$$

Perform 3 iterations with the power method in order to establish the smallest eigenvalue of B starting from the initial eigenvector $[1, 1, 1]^T$. Give your *final* eigenvalue approximation correct to 2 decimal places. ***(7 marks)***

2 (i) A polynomial of degree n can be expressed by the following sum,

$$P_n(x) = \sum_{j=0}^{n} a_j x^j.$$

In the least squares sense, a unique polynomial of degree n can be fitted to data points $(x_i, f(x_i))$, where $i = 0, 1, 2, \ldots, m$ and $m \geq n$. Assuming that the x_i values are free of errors, derive the normal equations

$$\sum_{i=0}^{m} \left(\sum_{j=0}^{n} a_j x_i^{j+k} \right) = \sum_{i=0}^{m} x_i^k f_i, \qquad k = 0, 1, 2, \ldots, n.$$

(6 marks)

(ii) For a spring experiment, the restoring force is given as F, in Newtons, and the distance the spring is stretched from the equilibrium position is given as x, in centimetres. The measured data is shown in the following table.

x (cm)	0	5	10	15	20	25	30
F (N)	0	-2.2	-5.4	-9.6	-14.8	-21.0	-28.2

Fit a least squares quadratic polynomial to this data giving the *final* coefficients to an accuracy of 2 decimal places.

Assuming Hooke's Law is valid for the same spring in the range $x \in [0, 5]$ cm calculate the spring constant k in N/cm. Using this value of k what does Hooke's Law predict the restoring force to be when $x = 7$ cm? Is this greater or less than (in magnitude) the restoring force estimated from the least squares quadratic fit when $x = 7$ cm and by how much? ***(19 marks)***

3 (i) The Lagrange interpolation polynomial of least degree which passes through $(n+1)$ points (x_i, f_i), $i = 0, 1, 2, \ldots, n$ is

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f_i$$

where

$$L_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

and $f_i = f(x_i)$. Given that the x_i values are tabulated at a constant interval h and using $x_i = x_0 + ih$, show that the Lagrange polynomial passing through 3 points (x_i, f_i), where $i = 0, 1, 2$ can be expressed as

$$P_2(x_i) = \frac{1}{2}\left[(i-1)(i-2)f_0 - 2i(i-2)f_1 + i(i-1)f_2\right]. \qquad (1)$$

Also show that

$$P_2'(x_1) = \frac{f_2 - f_0}{2h}, \qquad (2)$$

where $P_2' = dP_2/dx$.

(12 marks)

(ii) Use equation (1) from part (i) to derive Simpson's rule by integrating between x_0 and x_2.

(5 marks)

(iii) Given the incomplete table,

x	1	1.5	2	2.5	3
$f(x)$	1	1.7171	2.5198	3.3930	4.3267
$f'(x)$	1.3333	?	?	?	1.9230

use equation (2) from part (i) to estimate $f'(1.5)$, $f'(2)$ and $f'(2.5)$, then use the composite Simpson's rule to calculate the following integral,

$$\int_1^3 f'(x)dx.$$

(8 marks)

4 (i) A function $y(x)$ satisfies the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + e^x y = 0 \qquad (3)$$

and the conditions $y(0) = 1$ and $y'(0) = 0$. Use Euler's method with step size $h = 0.1$ to determine $y(0.4)$. Give your *final* answer to an accuracy of 5 decimal places.

Hint: Rewrite equation (3) as two coupled first order ordinary differential equations. ***(8 marks)***

(ii) Write down the iteration formula for the improved Euler's method. ***(2 marks)***

(iii) An engineering firm produces two types of products, P_1 and P_2, which require three types of raw materials, A, B and C, to manufacture. Each unit of P_1 uses 1 kilogram of A, 2 kilograms of B and 3 kilograms of C. Each unit of P_2 uses 2 kilograms of A and 1 kilogram of C. The raw materials available each day are 380 kilograms of A, 320 kilograms of B and 540 kilograms of C. The profit per unit of P_1 and P_2 is £15 and £12, respectively.

Formulate this into a linear programming problem and use graphical methods to determine the maximum possible daily profit. On the graph, clearly show the feasibility region and the line of constant revenue through the point of maximum daily profit.

(15 marks)

End of Question Paper