The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Probability Modelling **2 hours**

*Candidates should attempt **ALL** four questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–30; Q2–21; Q3–25; Q4–24)

1 A group of four badgers share two setts, A and B. Each day, exactly one badger moves from the sett it is currently in to the other one, each badger being equally likely to move. Let X_n be the number of badgers in sett A on day n, and model this as a Markov chain with state space $\{0, 1, 2, 3, 4\}$.

(a) Give the transition matrix of the Markov chain. ***(3 marks)***

(b) Show that the Markov chain is irreducible, and find the period of each state of the Markov chain. ***(6 marks)***

(c) Write down the equations for a stationary distribution of the Markov chain, and show that they are satisfied by $\begin{pmatrix} \frac{1}{16} & \frac{4}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} \end{pmatrix}$. ***(5 marks)***

(d) Assume that, on day 0, exactly two of the badgers are in sett A.

(i) Show by induction that for $n \geq 1$, then the row vector giving distribution of the state of the chain at time $2n - 1$ is $\begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$. ***(4 marks)***

(ii) Show that the distribution of the state of the chain at time n does not converge to the stationary distribution in (c) as $n \to \infty$. ***(4 marks)***

(e) For each possible starting state of the chain, find the probability that all the badgers are in sett A before they are all in sett B. ***(8 marks)***

2 In a non-delayed renewal process, let $f_n = P(T_i = n)$ where T_i is the ith inter-renewal time, and let u_n be the probability that a renewal takes place at time n. Consider such a process with $f_n = \dfrac{1}{3^n} - \dfrac{1}{4^n}$.

(a) Show that $u_1 = 1/12$ and $u_2 = 1/18$. ***(3 marks)***

(b) Find the generating function $F(s)$, defined as $\sum_{n=1}^{\infty} f_n s^n$ for $|s| < 1$. ***(3 marks)***

(c) Is the renewal process transient or recurrent? Explain your answer carefully. ***(3 marks)***

(d) Let $U(s)$ be the generating function of the sequence $(u_n)_{n\geq 0}$. Show that

$$U(s) = 1 + \frac{1}{2}\left(\frac{1}{2-s} - \frac{3}{6-s}\right).$$

(6 marks)

(e) Hence find an expression for u_n, and show that $u_n \to 0$ as $n \to \infty$. ***(6 marks)***

3 Let (X_n) be a Markov chain on $S = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Show that the chain is irreducible and aperiodic. ***(6 marks)***

(b) Deduce that, for all $i \in S$, $P(X_n = i)$ converges to a limit as $n \to \infty$, and give the value of the limit. ***(10 marks)***

(c) Find the expected number of steps until the chain returns to state 2,

(i) starting from state 1. ***(6 marks)***

(ii) starting from state 2. ***(3 marks)***

4 In a region of the Galaxy, the locations of star systems are modelled according to a spatial Poisson process with rate $\frac{1}{100}$ per cubic light year.

(a) Find the probability that there is exactly one star system within three light years of a specific point. [Reminder: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.] ***(4 marks)***

(b) Given that there are exactly five star systems within five light years of a specific point, find

(i) the probability that there are exactly eight star systems within six light years of the specific point. ***(4 marks)***

(ii) the probability that there are no star systems within three light years of the specific point. ***(4 marks)***

(c) Find the probability density function of the distance to the nearest star system from a specific point. ***(6 marks)***

(d) Each star system is assumed to contain a habitable planet with probability $\frac{1}{10}$. Making appropriate assumptions, which you should state clearly, find the probability that there is at least one star system containing a habitable planet within five light years of a specific point. ***(6 marks)***

End of Question Paper