1 A group of four badgers share two sets, A and B. Each day, exactly one badger moves from the set it is currently in to the other one, each badger being equally likely to move. Let $X_n$ be the number of badgers in set A on day $n$, and model this as a Markov chain with state space $\{0, 1, 2, 3, 4\}$.

(a) Give the transition matrix of the Markov chain. (3 marks)

(b) Show that the Markov chain is irreducible, and find the period of each state of the Markov chain. (6 marks)

(c) Write down the equations for a stationary distribution of the Markov chain, and show that they are satisfied by $\begin{pmatrix} \frac{1}{16} & \frac{4}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} \end{pmatrix}$. (5 marks)

(d) Assume that, on day 0, exactly two of the badgers are in set A.

(i) Show by induction that for $n \geq 1$, then the row vector giving distribution of the state of the chain at time $2n-1$ is $\begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$. (4 marks)

(ii) Show that the distribution of the state of the chain at time $n$ does not converge to the stationary distribution in (c) as $n \to \infty$. (4 marks)

(e) For each possible starting state of the chain, find the probability that all the badgers are in set A before they are all in set B. (8 marks)
In a non-delayed renewal process, let \( f_n = P(T_i = n) \) where \( T_i \) is the \( i \)th inter-renewal time, and let \( u_n \) be the probability that a renewal takes place at time \( n \). Consider such a process with \( f_n = \frac{1}{3^n} - \frac{1}{4^n} \).

(a) Show that \( u_1 = 1/12 \) and \( u_2 = 1/18 \). \((3 \text{ marks})\)

(b) Find the generating function \( F(s) \), defined as \( \sum_{n=1}^{\infty} f_n s^n \) for \(|s| < 1\). \((3 \text{ marks})\)

(c) Is the renewal process transient or recurrent? Explain your answer carefully. \((3 \text{ marks})\)

(d) Let \( U(s) \) be the generating function of the sequence \((u_n)_{n \geq 0}\). Show that

\[
U(s) = 1 + \frac{1}{2} \left( \frac{1}{2 - s} - \frac{3}{6 - s} \right).
\]

\((6 \text{ marks})\)

(e) Hence find an expression for \( u_n \), and show that \( u_n \to 0 \) as \( n \to \infty \). \((6 \text{ marks})\)

Let \((X_n)\) be a Markov chain on \( S = \{1, 2, 3, 4\} \) with transition matrix

\[
P = \begin{pmatrix}
0 & 1 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

(a) Show that the chain is irreducible and aperiodic. \((6 \text{ marks})\)

(b) Deduce that, for all \( i \in S \), \( P(X_n = i) \) converges to a limit as \( n \to \infty \), and give the value of the limit. \((10 \text{ marks})\)

(c) Find the expected number of steps until the chain returns to state 2,

(i) starting from state 1. \((6 \text{ marks})\)

(ii) starting from state 2. \((3 \text{ marks})\)
In a region of the Galaxy, the locations of star systems are modelled according to a spatial Poisson process with rate \( \frac{1}{100} \) per cubic light year.

(a) Find the probability that there is exactly one star system within three light years of a specific point. [Reminder: the volume of a sphere of radius \( r \) is \( \frac{4}{3}\pi r^3 \).] \( (4 \text{ marks}) \)

(b) Given that there are exactly five star systems within five light years of a specific point, find

(i) the probability that there are exactly eight star systems within six light years of the specific point. \( (4 \text{ marks}) \)

(ii) the probability that there are no star systems within three light years of the specific point. \( (4 \text{ marks}) \)

(c) Find the probability density function of the distance to the nearest star system from a specific point. \( (6 \text{ marks}) \)

(d) Each star system is assumed to contain a habitable planet with probability \( \frac{1}{10} \). Making appropriate assumptions, which you should state clearly, find the probability that there is at least one star system containing a habitable planet within five light years of a specific point. \( (6 \text{ marks}) \)

End of Question Paper