The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2016-2017**

**Mechanics and Fluids** **2 hours**

*Attempt all four questions. The allocation of marks is shown in brackets.*

**1** (i) Let

$$\phi = x^3 + y\,.$$

Calculate $\nabla\phi$. ***(1 mark)***

(ii) For the $\phi$ of part (i) sketch isocontours (level curves) for positive $\phi$. Add at least two vectors indicating the direction of $\nabla\phi$. ***(4 marks)***

(iii) Also for the $\phi$ of part (i), calculate the rate of change of $\phi$ in the direction $(4,3)$ at position $(2,3)$. ***(3 marks)***

(iv) Let a force $\boldsymbol{F}$ be a conservative force defined by the scalar potential $\phi$ of part (i). Determine the force. ***(2 marks)***

(v) Without resorting to a path integral, what is the work done by the potential force of the previous part in going from position $(4,2)$ to $(2,0)$. You may assume that all quantities are given in standard units. ***(2 marks)***

(vi) Consider three point bodies of mass 3 kg, 4 kg and 5 kg at positions $(3,3)$, $(1,2)$ and $(7,3)$ respectively, where lengths are measured in metres. Find the position of their centre of mass. ***(3 marks)***

(vii) Let

$$\boldsymbol{F} = 2xz\boldsymbol{i} + x^2\boldsymbol{k}\,.$$

Calculate $\nabla \cdot \boldsymbol{F}$. ***(2 marks)***

(viii) Calculate $\nabla \times \boldsymbol{F}$ for the force of part (vii). ***(2 marks)***

(ix) Also for the force of part (vii), calculate $(\boldsymbol{F} \cdot \nabla)\boldsymbol{F}$. ***(3 marks)***

(x) Simplify the expression $\varepsilon_{1jk}\,\delta_{j3}\,a_k$ as far as possible. What is the relationship between this expression and $(\boldsymbol{k} \times \boldsymbol{a})$? ***(3 marks)***

**2** (i) A particle of mass $m$ follows a path described in polar coordinates by $r = r(t)$ and $\theta = \theta(t)$. Under the influence of a central force $F(r)$ about the origin, its motion is described by the equation

$$(\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\hat{\boldsymbol{\theta}} = -F(r)\,\hat{\boldsymbol{r}}\,.$$

Given that $\dot{\boldsymbol{r}} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$, find an expression for the specific angular momentum $\boldsymbol{h}$. Hence show that angular momentum of the mass about the origin is conserved. ***(8 marks)***

(ii) A meteorologist constructs a weather vane from a right-angled triangle laminar with perpendicular sides of length $2a$ and $a$. It rotates about an axis through the middle of, and perpendicular to, the side of length $2a$. If the area density of the laminar is given by $\sigma = (2a - x)\,\sigma_0/a$, where $x$ is the distance from the axis, determine the position of its centre of mass by considering vertical strips. Find also the moment of inertia of the laminar about its axis of rotation.

The meteorologist tests the weather vane by applying a force of $0.02\,\mathrm{N}$ for $0.5\,\mathrm{s}$ to the tip of the triangle, which is initially at rest. The force is in the direction of rotation, causing it to spin. Neglecting resistance, if $a = 0.2\,\mathrm{m}$ and $\sigma_0 = 15\,\mathrm{kg\,m^{-2}}$, how fast does it spin after it is pushed? ***(17 marks)***

**3** (i) Let $r = |\boldsymbol{r}|$, where $\boldsymbol{r} = (x_1, x_2, x_3)$. Show that

$$\frac{\partial r}{\partial x_1} = \frac{x_1}{r}\,.$$

*Using suffix notation,* show that the following holds for any vector $\boldsymbol{A}$:

$$\boldsymbol{\nabla} \times (r\,\boldsymbol{A}) \;=\; r\,\boldsymbol{\nabla} \times \boldsymbol{A} \;-\; \frac{1}{r}\,\boldsymbol{A} \times \boldsymbol{r}\,.$$

***(9 marks)***

(ii) A cylindrical section is defined on the surface $r = a$ with limits $0 \le \theta \le \pi/2$, $0 \le z \le h$. Sketch the surface.

State Stokes' theorem and show that it holds for the case $\boldsymbol{F} = yz\,\boldsymbol{i}$ on the cylindrical section. *Hint:* Write an expression in Cartesian coordinates for the $\hat{\boldsymbol{r}}$ of cylindrical coordinates. ***(16 marks)***

**4** The flow *inside* a cylinder of radius $a$ is given by the stream function

$$\psi = U\,(r + \gamma r^2)\,\sin\theta\,,$$

where $U$ and $\gamma$ are constants, for a fluid of density $\rho$. In cylindrical polar coordinates $(r, \theta, z)$ the flow may be written

$$\boldsymbol{u} = \boldsymbol{\nabla} \times (\psi \hat{\boldsymbol{z}}) = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\,\hat{\boldsymbol{r}} - \frac{\partial \psi}{\partial r}\,\hat{\boldsymbol{\theta}},$$

(i) Show that the velocity field $\boldsymbol{u}$ satisfies the incompressibility condition.
Find $\boldsymbol{u}$.
Use the appropriate boundary condition on $r = a$ to determine $\gamma$.
Find $\boldsymbol{u}$ at the origin and convert your result into Cartesian coordinates.

***(8 marks)***

(ii) Find the stagnation points and sketch the flow.

***(9 marks)***

(iii) Given that the pressure at the origin is $p_0$, find an expression for the pressure $p$ on the boundary surface.
The total force over the boundary surface $S$ is given by

$$\boldsymbol{F} \;=\; -\int_S p\,\mathrm{d}\boldsymbol{S}\,.$$

Show that the $x$-component of the force is zero.

***(8 marks)***

## End of Question Paper