The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Fluid Mechanics I **2 hours**

Answer all four questions.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

1 (i) Write down the Navier-Stokes (N-S) equation for an incompressible fluid of velocity $\mathbf{v}$ with constant density under the influence of the gravitational field, and give a brief explanation of the terms in the N-S equation. State how to obtain the Euler equation for the ideal fluid. Discuss the criterion that determines whether the flow is laminar or turbulent.

(7 marks)

(ii) State the physical meaning of slip-free boundary conditions.

(2 marks)

(iii) Water flows through a 100-mm diameter pipe at the constant velocity of 3.00m/s through the cross-section. Find the volume flow rate (volume flux) and mass flow rate (mass flux) in MKS units. The density of water is 1g/cm^3.

(6 marks)

(iv) Given the time-dependent velocity field $\mathbf{v}(x, y, z, t) = (3t, xz, ty^2)$, find the acceleration of a particle. Here t denotes time.

(5 marks)

(v) Prove the following identity for a general velocity field $\mathbf{v}$

$$\nabla^2 \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times \boldsymbol{\omega}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$.

(5 marks)

2 (i) Consider a fluid element inside a small circle normal to the local vorticity $\boldsymbol{\omega}$. By assuming that translation and pure straining motion of this fluid element can be ignored, derive the relation between $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ and the angular velocity of the fluid.

(7 marks)

(ii) Hence, determine the angular velocity vector of a fluid particle at $(2, 3, 4)$ for the given velocity field $\mathbf{v} = (13x^2y, 18(yz + x), 15)$.

(5 marks)

(iii) Consider an incompressible fluid inside a small sphere of radius a. If the fluid has stress tensor σ_{ij}, find the average value of the normal component of stress over the surface of this small sphere by using two different methods (i.e. by using isotropy and by computing directly in the spherical coordinates). Assume that σ_{ij} is constant inside the sphere.

(13 marks)

3 Consider two thin cylindrical shells with the same vertical axis. Let the inner and outer shells be of radius r_1 and r_2, respectively. Suppose that the annular region $r_1 \leq r \leq r_2$ is filled with incompressible fluid of density ρ and viscosity μ. Let the inner and outer cylinders rotate at the constant angular velocities Ω_1 and Ω_2, respectively. In cylindrical polar coordinates (r, θ), the Navier-Stokes equations in the absence of body forces are

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right),$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right),$$

and the continuity equation is

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0,$$

where v_r and v_θ are the velocity components in the r and θ directions. In parts (i)-(iii) below you can use this identity:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Assume that the flow is steady and suppose that the flow velocity within the fluid is

$$\mathbf{v} = v(r)\boldsymbol{e}_\theta = r\Omega(r)\boldsymbol{e}_\theta,$$

where $\Omega(r) = v(r)/r$ is the angular velocity profile. Here, r denotes distance from the axis and $\boldsymbol{e}_\theta$ is a unit vector in the direction of increasing θ. You may assume that body forces are negligible.

(i) Show that $\nabla \cdot \mathbf{v} = 0$ ***(2 marks)***

(ii) Show that Ω satisfies

$$\frac{1}{r^2}\frac{d}{dr}\left(r^3\frac{d\Omega}{dr}\right) = 0.$$

Hence, show that the pressure distribution depends only upon r.

(13 marks)

(iii) You are given that the solution to part (ii) is

$$\Omega(r) = \frac{1}{r_2^2 - r_1^2}\left(\frac{r_1^2 r_2^2(\Omega_1 - \Omega_2)}{r^2} + (r_2^2\Omega_2 - r_1^2\Omega_1)\right).$$

Show that this solution satisfies the boundary conditions. Also, compute all non-zero components of the tangential stress tensor within the fluid. Hence compute the torque from the viscous stress acting in the θ direction per unit height exerted by the fluid on the inner cylinder. You can use the following expression for the deformation tensor e_{ij} in (r, θ) coordinates:

$$e_{rr} = \frac{\partial v_r}{\partial r}, \; e_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}, \; e_{r\theta} = \frac{1}{2r}\frac{\partial v_r}{\partial \theta} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right).$$

(10 marks)

4 A vertical tube of circular cross-section, infinite length and uniform internal radius a is filled with incompressible fluid with velocity $\mathbf{v}$. The tube moves vertically upwards with the constant speed U. You are given that the velocity field is of the form

$$\mathbf{v} = w(r)\boldsymbol{e}_z$$

and that the pressure gradient in the fluid in the vertical direction is zero. In cylindrical polar coordinates (r, θ, z) you can use the following identities:

$$\mathbf{v} \cdot \nabla = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z},$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

(i) Show that

$$w = U + \frac{g}{4\nu}(r^2 - a^2),$$

where g is the acceleration due to gravity and ν is the kinematic viscosity.

(15 marks)

(ii) If flow is such that the net flux of fluid in the tube across any fixed horizontal plane is zero, show that

$$U = \frac{ga^2}{8\nu}.$$

Hence compute the net momentum flux of fluid per unit mass in the tube across any fixed horizontal plane.

(10 marks)

End of Question Paper